Some proposals about VFE discretization in NH modeling

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Outline

- Introduction
- Vertical Finite Elements and discrete linear operators
- Vertical Laplacian and its eigenvalues
- Conclusions

Introduction

Previous work and context

Vertical Finite Elements

- Operational VFE in hydrostatic IFS (A. Untch and M. Hortal)
- Studies about VFE in ALADIN community (P. Bènard, J. Vivoda and others)

ALADIN dynamical core

- Semi-Implicit Semi-Lagrangian Spectral NH model
- Mass based vertical coordinate

Stability studies

- SBH stability, prognostic variables dependence

And more documentation and papers form ECWMF and ALADIN

EE in terrain-following hydrostatic-pressure coordinate (Laprise,1992)

Prognostic equations

$$\frac{d\boldsymbol{V}}{dt}+RT\nabla_\eta \ln p+\frac{1}{m}\frac{\partial p}{\partial \eta}\nabla_\eta \phi=\boldsymbol{F}$$

$$\gamma\frac{dw}{dt}+g\left(1-\frac{1}{m}\frac{\partial p}{\partial \eta}\right)=\gamma F_z$$

$$\frac{dT}{dt}-\frac{RT}{c_p}\frac{d\ln p}{dt}=\frac{Q}{c_p}$$

$$\frac{d\ln p}{dt}+\frac{c_p}{c_v}(D+d)=\frac{Q}{c_v T}$$

$$\frac{\partial m}{\partial t}+\nabla_\eta\cdot(m\boldsymbol{V})+\partial_\eta(\dot{\eta}m)=0$$

alternative

$$\frac{d\phi}{dt}-wg=0$$

where

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\boldsymbol{V}\cdot\nabla_\eta+\dot{\eta}\frac{\partial}{\partial \eta}$$

$$D=\nabla_\eta\cdot\boldsymbol{V}+\frac{1}{m}\rho\nabla_\eta\phi\cdot\frac{\partial \boldsymbol{V}}{\partial \eta}$$

$$d=-\frac{1}{m}\rho g\frac{\partial w}{\partial \eta}$$

Vertical coordinate

$$\frac{\partial \pi}{\partial z}=-\rho g$$

$$\pi(x,y,\eta,t)=A(\eta)+B(\eta)\pi_s(x,y,t)$$

$$m=\frac{\partial \pi}{\partial \eta}$$

Semi-implicit time discretization

$$\frac{d\boldsymbol{X}}{dt}=\boldsymbol{M}(\boldsymbol{X},t)$$

Use a linear model implicitly for stability

$$\frac{\boldsymbol{X}^{t+\Delta t}-\boldsymbol{X}^{t}}{\Delta t}=\boldsymbol{M}(\boldsymbol{X}^{t+\Delta t/2})+\boldsymbol{L}^{*}(\frac{\boldsymbol{X}^{t+\Delta t}+\boldsymbol{X}^{t}}{2}-\boldsymbol{X}^{t+\Delta t/2})$$

- Advection terms can be Eulerian or Semi-Lagrangian
- For efficiency linear model must be horizontally homogeneous in model levels and time independent
- The reference atmosphere for the linear model is an isothermal T^* and hydrostatic atmosphere at rest
- Linear model is solved in spectral space. It is found a vertically coupled linear system in one variable for each eigenfuction of the horizontal Laplacian

ALADIN Linear System (scientific documentation)

Prognostic equations

$$\frac{\partial D}{\partial t} = -R_d G^* \Delta T' + R_d T^* (G^* - 1) \Delta P - \frac{R_d T^*}{\pi_s^*} \Delta \pi_s' - \Delta \phi_s'$$

$$\frac{\partial T'}{\partial t} = -\frac{R_d T^*}{c_v}(D + d) \quad \text{temperature}$$

$$\frac{\partial \pi_s'}{\partial t} = -\pi_s^* N^* D \quad \text{hydrostatic pressure at surface}$$

$$\frac{\partial d}{\partial t} = -\frac{g^2}{R_d T^*} L^* P \quad \text{vertical divergence}$$

$$\frac{\partial P}{\partial t} = S^* D - \frac{c_p}{c_v}(D + d) \quad \text{pressure departure}$$

Constraint

$$G^* + S^* - G^* S^* - N^* = 0$$

Vertical coordinate

$$\eta = \frac{\pi^*(\eta)}{\pi_s^*}$$

Vertical Linear operators

$$N^* X(\eta) = \int_0^1 X(\eta)\, d\eta$$

$$G^* X(\eta) = \int_\eta^1 X \frac{(\eta')}{\eta'}\, d\eta'$$

$$S^* X(\eta) = \frac{1}{\eta} \int_0^\eta X(\eta')\, d\eta'$$

$$\tilde{\partial} = \eta \frac{\partial}{\partial \eta}$$

$$L^* = (\tilde{\partial} + 1)\tilde{\partial}$$

Linear system without integral operators (Agathe Untch)

Prognostic equations

$$\frac{\partial D}{\partial t} = -\Delta \phi' - \frac{R_d T^*}{\pi^*} \Delta p' \quad \text{horizontal divergence}$$

$$\frac{\partial w}{\partial t} = -g \left(\frac{m'}{m^*} - \frac{1}{m^*} \frac{\partial p'}{\partial \eta} \right) \quad \text{vertical velocity}$$

$$\frac{\partial p'}{\partial t} = -\frac{c_p}{c_v} \pi^* D + g \frac{c_p}{c_v} \frac{\pi^{*2}}{R_d T^* m^*} \frac{\partial w}{\partial \eta} \quad \text{pressure}$$

$$\frac{\partial m'}{\partial t} = -m^* \left(D + \frac{\partial \dot{\eta}'}{\partial \eta} \right) \quad \text{factor } m$$

$$\frac{\partial \phi'}{\partial t} = w g \quad \text{geopotential}$$

Vertical coordinate

$$\eta = \frac{\pi^*(\eta)}{\pi_s^*}$$

Vertical Linear operators

$$\tilde{\partial} = \eta \frac{\partial}{\partial \eta}$$

$$L^* = (\tilde{\partial} + 1) \tilde{\partial}$$

Linear system with P, π_s instead p, m (Juan Simarro)

Prognostic equations

$$\frac{\partial D}{\partial t} = \frac{-R_d T^*}{\pi^*}(B \Delta \pi_s' + \pi^* \Delta P) - \Delta \phi' \text{ horizontal divergence}$$

$$\frac{\partial w}{\partial t} = g(1+\tilde{\partial})P \text{ vertical velocity}$$

$$\frac{\partial P}{\partial t} = -\frac{c_p}{c_v}\left(D - \frac{g}{R_d T^*}\tilde{\partial} w\right) + S^* D \text{ pressure departure}$$

$$\frac{\partial \pi_s'}{\partial t} = -\pi_s^* N^* D \text{ hydrostatic pressure at surface}$$

$$\frac{\partial \phi'}{\partial t} = w g + \frac{R_d T^*}{\pi^*}(B \pi_s^* N^* - \pi^* S^*) D \text{ geopotential}$$

Constraint

$$(1+\tilde{\partial}) S^* - I = 0$$

Vertical coordinate

$$\eta = \frac{\pi^*(\eta)}{\pi_s^*}$$

Vertical Linear operators

$$N^* X(\eta) = \int_0^1 X(\eta)\, d\eta$$

$$S^* X(\eta) = \frac{1}{\eta}\int_0^\eta X(\eta')\, d\eta'$$

$$\tilde{\partial} = \eta \frac{\partial}{\partial \eta}$$

$$L^* = (\tilde{\partial}+1)\tilde{\partial}$$

SHB stable linear system with $D, w, T, \phi, q = \ln \pi_s$ (Pierre Bènard)

Prognostic equations in σ coordinates	Vertical coordinate
$$\frac{\partial D}{\partial t} = -R_d \Delta T' - R_d T^* \Delta q' - (1+\tilde{\partial}) \Delta \phi'$$ divergence	$$\sigma = \frac{\pi^*(\sigma)}{\pi_s^*}$$
$$\frac{\partial w}{\partial t} = \frac{g}{R_d T^*} (R_d (1+\tilde{\partial}) T' + L^* \phi')$$ vertical velocity	Vertical Linear operators
$$\frac{\partial T'}{\partial t} = -\frac{R_d T^*}{c_v} (D - \frac{g}{R_d T^*} \tilde{\partial} w)$$ temperature	$$N^* X(\sigma) = \int_0^1 X(\sigma) d\sigma$$
$$\frac{\partial q'}{\partial t} = -N^* D$$ hydrostatic pressure at surface	$$S^* X(\sigma) = \frac{1}{\sigma} \int_0^\sigma X(\sigma') d\sigma'$$
$$\frac{\partial \phi'}{\partial t} = w g + R_d T^* (N^* - S^*) D$$ geopotential	$$\tilde{\partial} = \sigma \frac{\partial}{\partial \sigma}$$
Similar system in η coordinate	$$L^* = (\tilde{\partial} + 1) \tilde{\partial}$$
There are also some constraints, operator G^* not used	

Summary of linear systems

Linear system with different sets of prognostic variables

- Most of them use vertical integral and derivative operators
- It is proposed a NH linear system which do not use vertical integral operator
- Stability depends strongly on prognostic variables (P. Bènard, R. Laprise, J. Vivoda, and P. Smolkov). Some linear systems are SHB unstable
- Some continuous identities involving vertical opeartors are constraints for the vertical discretization. They must be satisfied in the discretization

Structure equation for one prognostic variable

- This equation represents acoustic and gravity waves which are present in NH modeling. For a 2-TL or 3-TL time Helmholtz equation is

$$(I-\delta t^2 c^{*2}(\Delta+\frac{L^*}{H^{*2}})-\delta t^4 N^{*2} c^{*2}\Delta)\chi=\hat{\chi}$$

- For stability reasons, vertical Laplacian L^* should have real and negative eigenvalues. In ALADIN this condition is different due to the presence of a tridiagonal matrix in the last term on the LHS

Vertical Finite Elements and discrete linear operators

In IFS model, VFE gives more accurate phase speeds of most of the linear gravity waves than FD, with a reduction of the vertical noise in forecasts

No staggering of dependent variables in physical space, making the method well suited for use with semi-Lagrangian advection

In NH modeling both integral and derivative operators are used in the linear and non linear model

The same VFE technique is used for integral and derivative operators

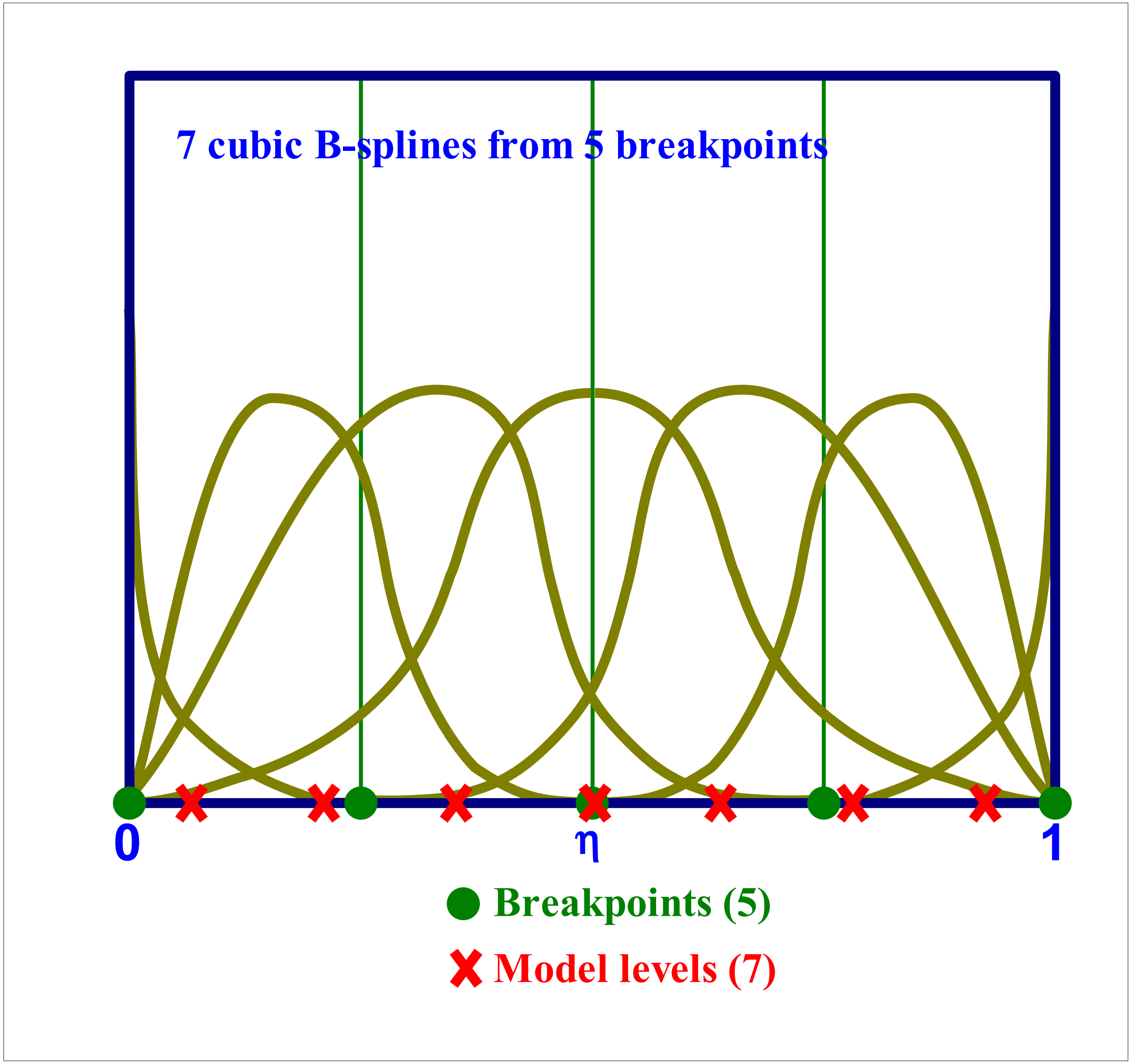
Method proposed, slightly different

- Same number of splines as model levels
- More straightforward implementation for both integral and derivative
- Valid for B-splines of any order
- Cubic spline integral operator has similar accuracy than IFS VFE integral

VFE and B-Splines

- Here a B-spline of degree k is any piecewise polynomial of degree k defined for $0 \leqslant \eta \leqslant 1$ such that it is continuous up to its $k-1$ derivative in a set of breakpoints $\{\zeta_{1,} \ldots, \zeta_P\}$
- Any B-spline of degree k can be expressed by a linear combination of a set of basis functions $\{N_1^k(\eta), \ldots, N_{P+k-1}^k(\eta)\}$. There is a recursive formula to obtain the set of basis functions (de Boor)
- The number of basis functions is chosen to be the number of model full levels, that is $N = P + k - 1$. Therefore the number of breakpoints depends on k and N and it is $P = N - k + 1$
- For linear, quadratic and cubic B-splines the number of breakpoints are N , $N-1$ and $N-2$ respectively
- Breakpoints $\{\zeta_{1,} \ldots, \zeta_P\}$ can be chosen independently from model full levels $\{\eta_{1,} \ldots, \eta_N\}$ although there are some restrictions

VFE and B-Splines



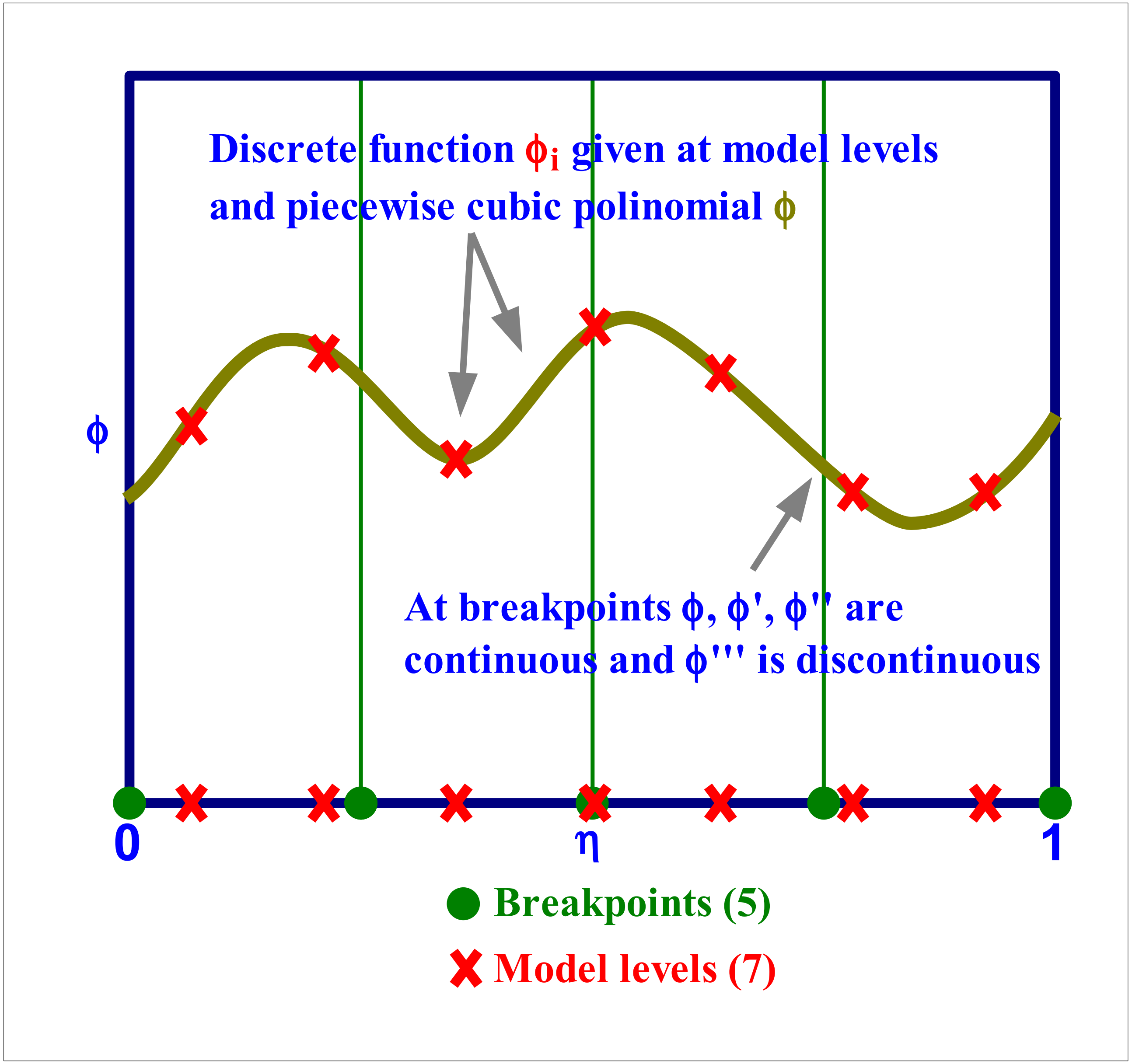
From full levels space to B-spline space and vice versa

- Given any discrete function $\{\varphi_1, \ldots, \varphi_N\}$ at full levels find real values $\{\hat{\varphi}_1, \ldots, \hat{\varphi}_N\}$ such that

$$\varphi_i = \sum_{j=1}^{N} \hat{\varphi}_j N_j^k(\eta_i)$$

- In matrix form $\boldsymbol{\varphi} = \boldsymbol{N} \cdot \hat{\boldsymbol{\varphi}}$ and then $\hat{\boldsymbol{\varphi}} = \boldsymbol{N}^{-1} \cdot \boldsymbol{\varphi}$
- Extended matrices like in IFS VFE scheme are not necessary here because full level space and B-spline space have the same dimension

From model levels to cubic B-spline space



VFE and Linear Operators

- Garlekin Method is applied in spline space
- Discrete version of any linear operator B is the $N\times N$ matrix

$$\boldsymbol{B}=\boldsymbol{N}\cdot\hat{\boldsymbol{M}}^{-1}\cdot\hat{\boldsymbol{B}}\cdot\boldsymbol{N}^{-1}$$

- where matrices $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{B}}$ are

$$\hat{M}_{ij}=\int_0^1 N_i^k(\eta)N_j^k(\eta)\,d\eta \text{ and } \hat{B}_{ij}=\int_0^1 N_i^k(\eta)\,B\,N_j^k(\eta)\,d\eta$$

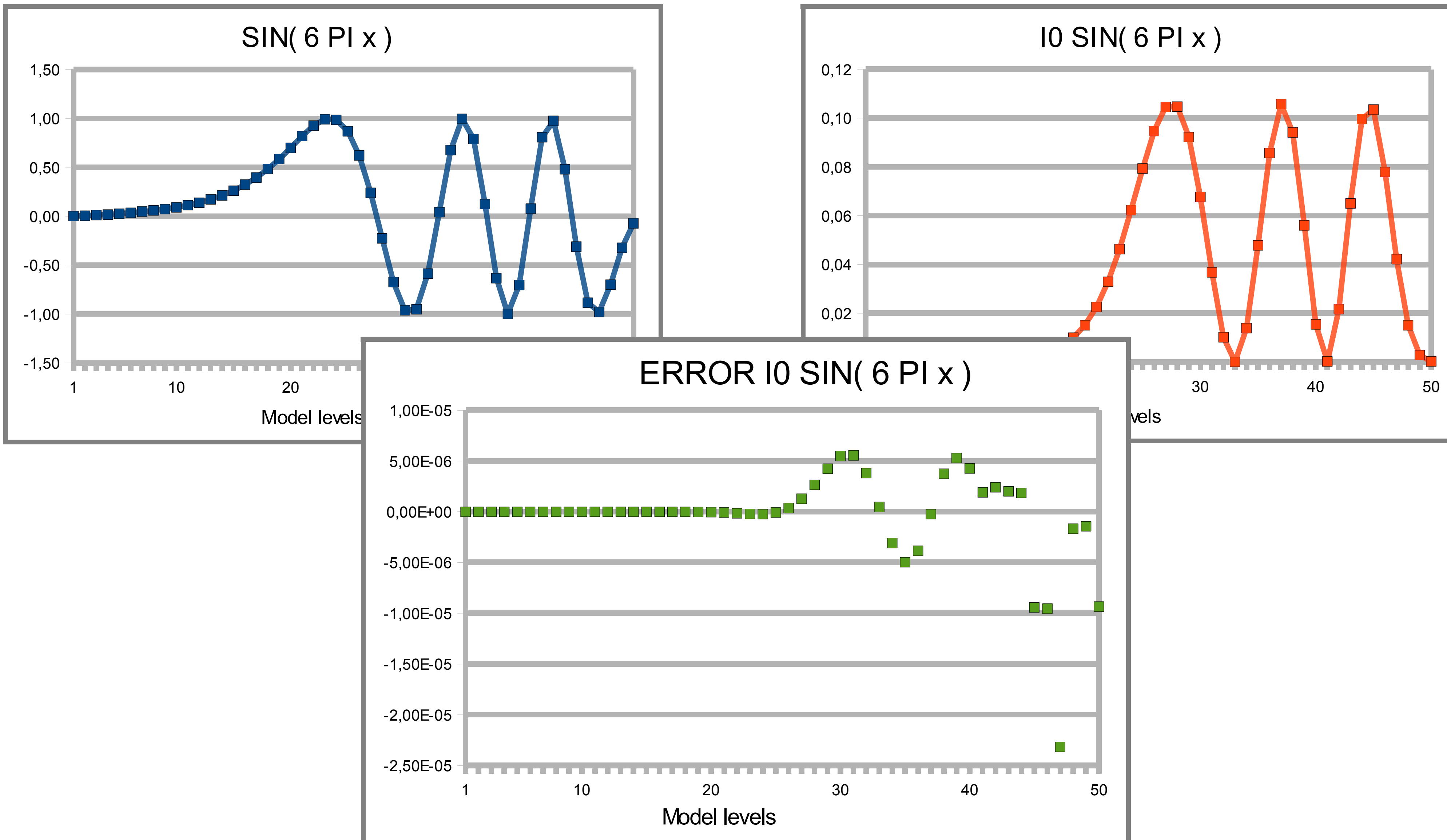
- The linear operators that appear in the linear model are

$$N^*X(\eta)=\int_0^1 X(\eta)\,d\eta\ ,\ G^*X(\eta)=\int_\eta^1 X\frac{(\eta')}{\eta'}\,d\eta'\ ,$$

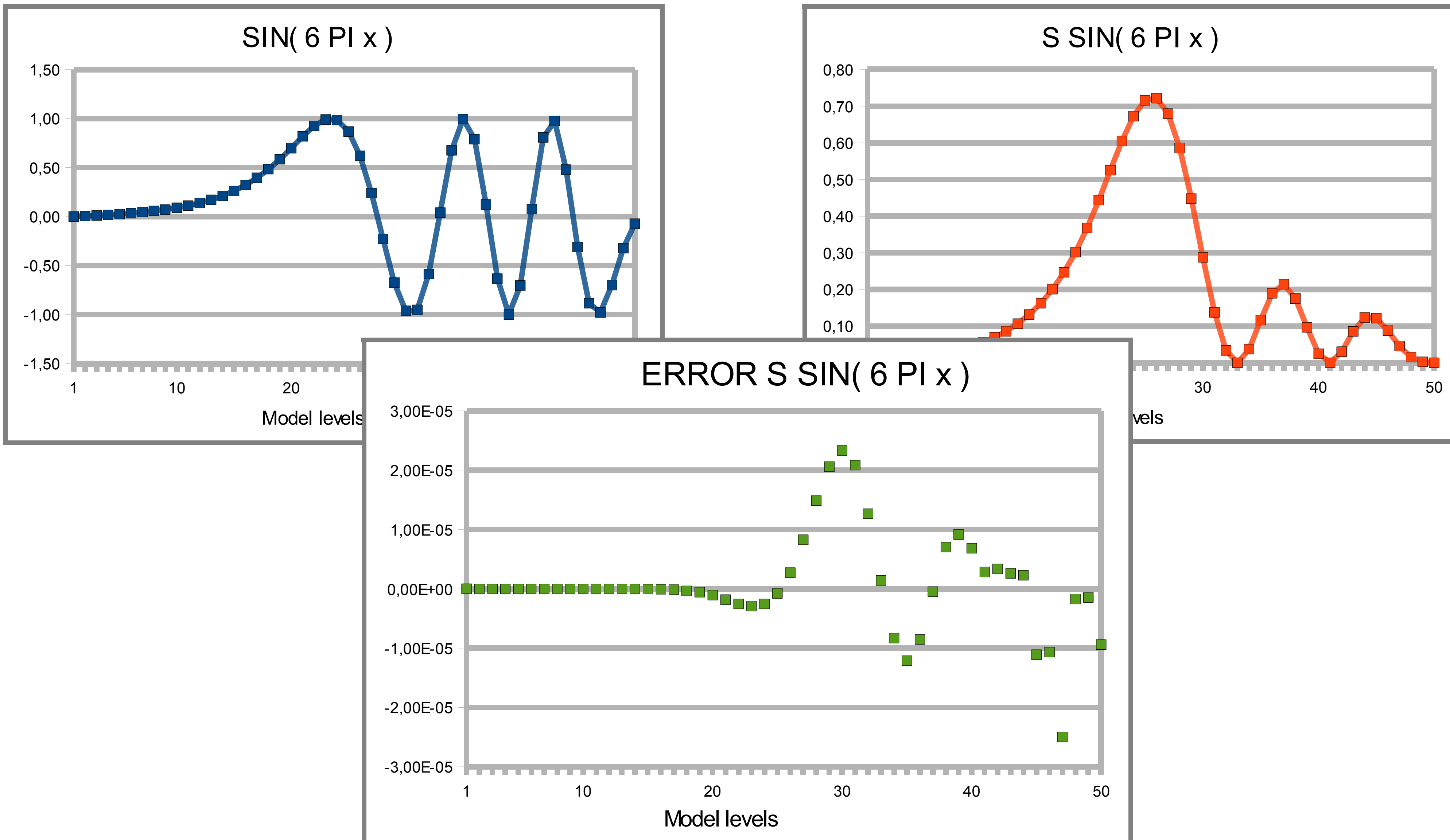
$$S^*X(\eta)=\frac{1}{\eta}\int_0^\eta X(\eta')\,d\eta'\ ,\ \tilde{\partial}=\eta\frac{\partial}{\partial\eta}\ ,\ L=(\tilde{\partial}+1)\tilde{\partial}$$

- For tests 50 full levels model from ECMWF and definition $\eta=\frac{\pi^*(\eta)}{\pi_s^*}$ and compare analytical and numerical results for $\varphi(\eta)=\sin(6\pi\eta)$

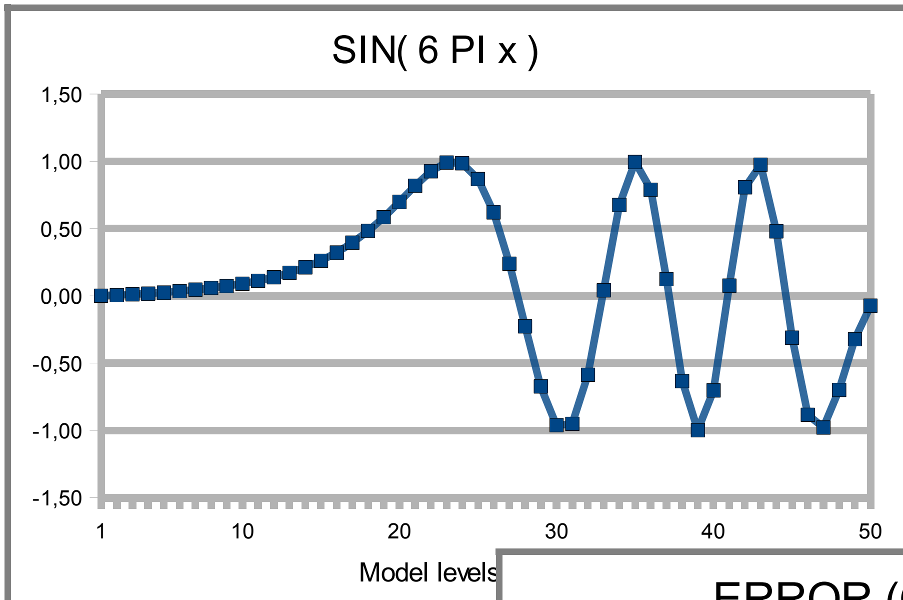
Test $I0\, X(\eta) = \int_0^{\eta} X(\eta)\, d\eta$

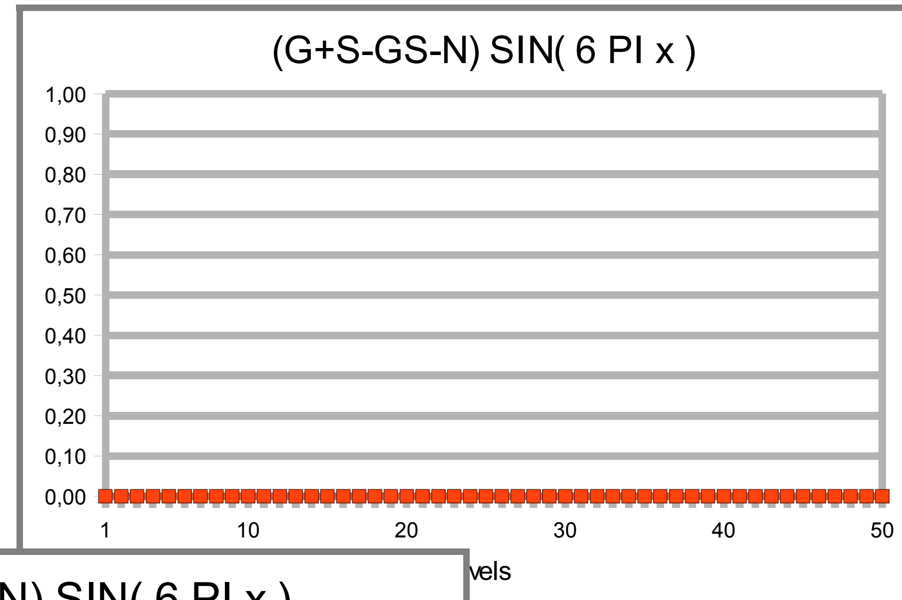


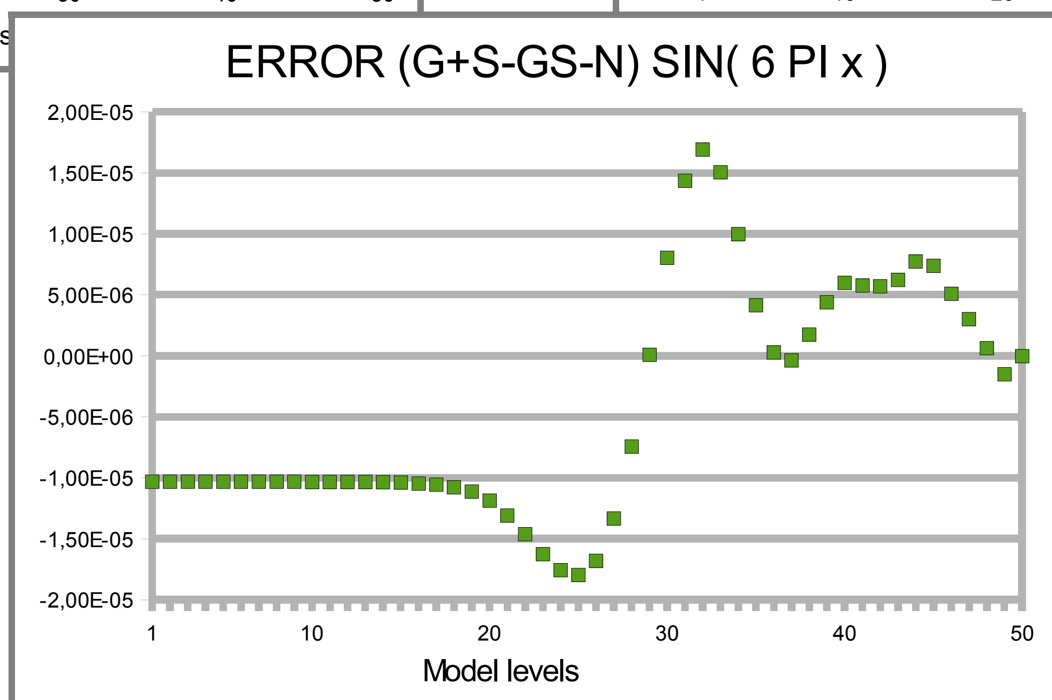
Test $S^{*}X(\eta)=\frac{1}{\eta}\int_{0}^{\eta}X(\eta')\,d\eta'$ matrix multiplication of $\boldsymbol{S=1/\eta\cdot I0}$



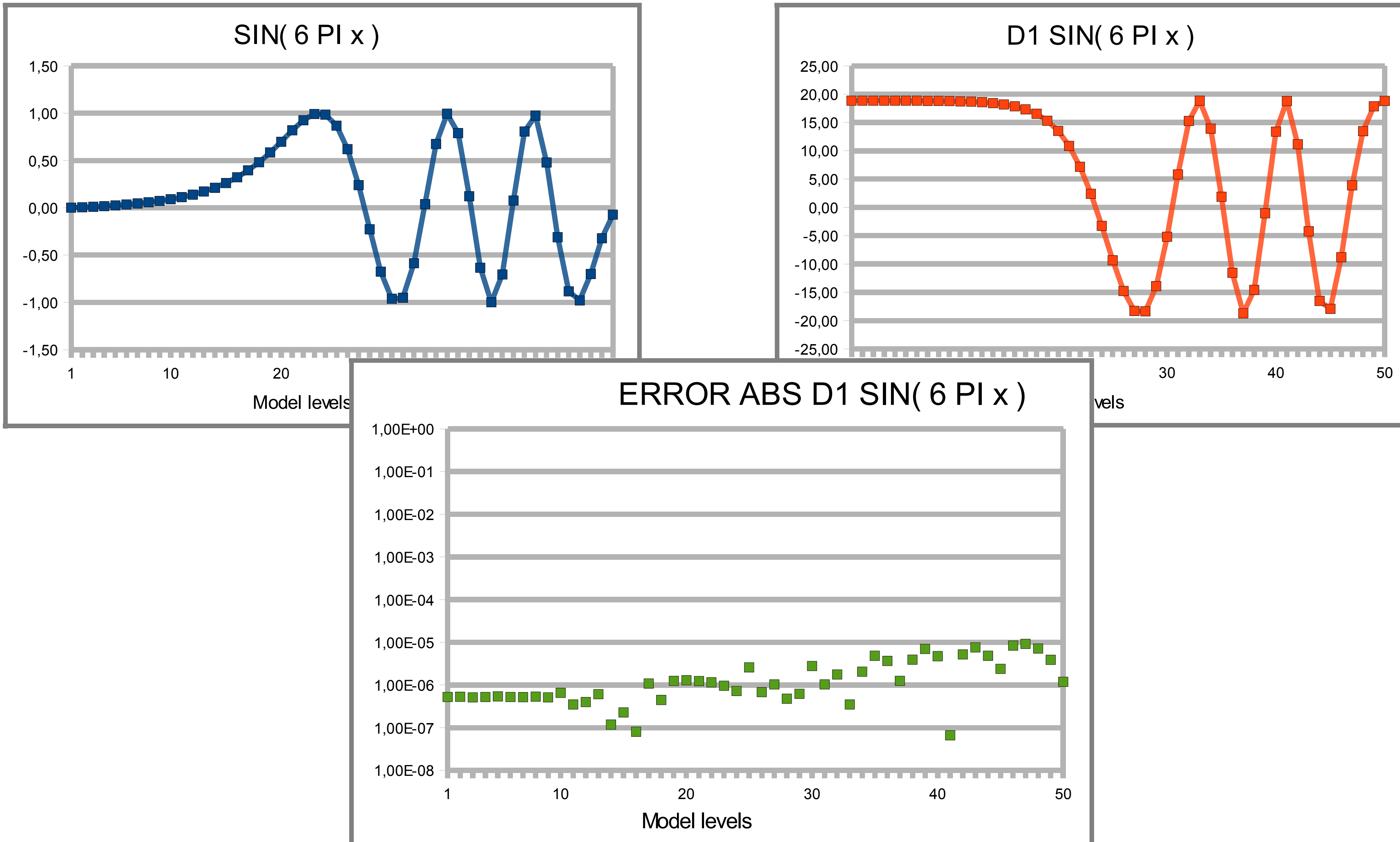
Test ALADIN C1 constraint $(G^{*}+S^{*}-G^{*}S^{*}-N^{*})\,X(\eta)$



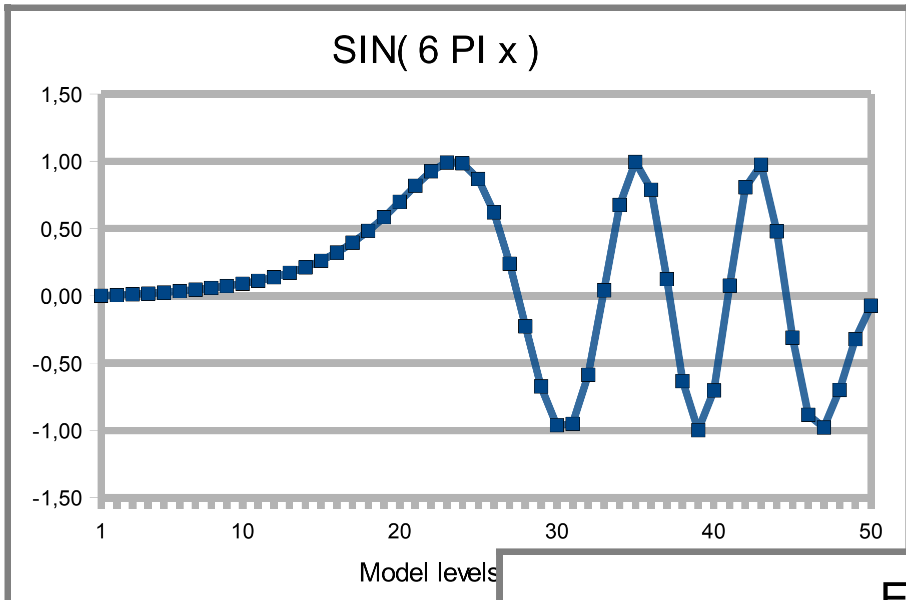


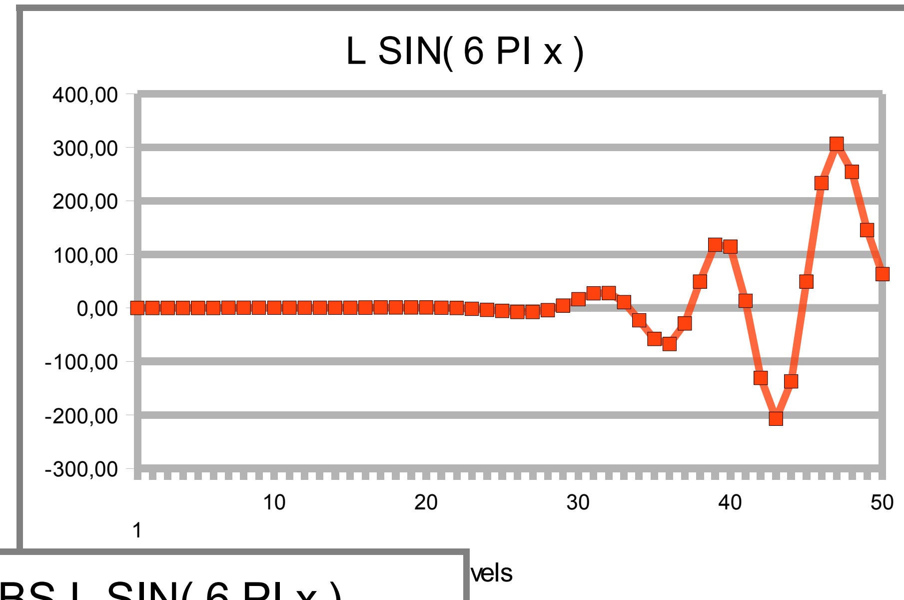


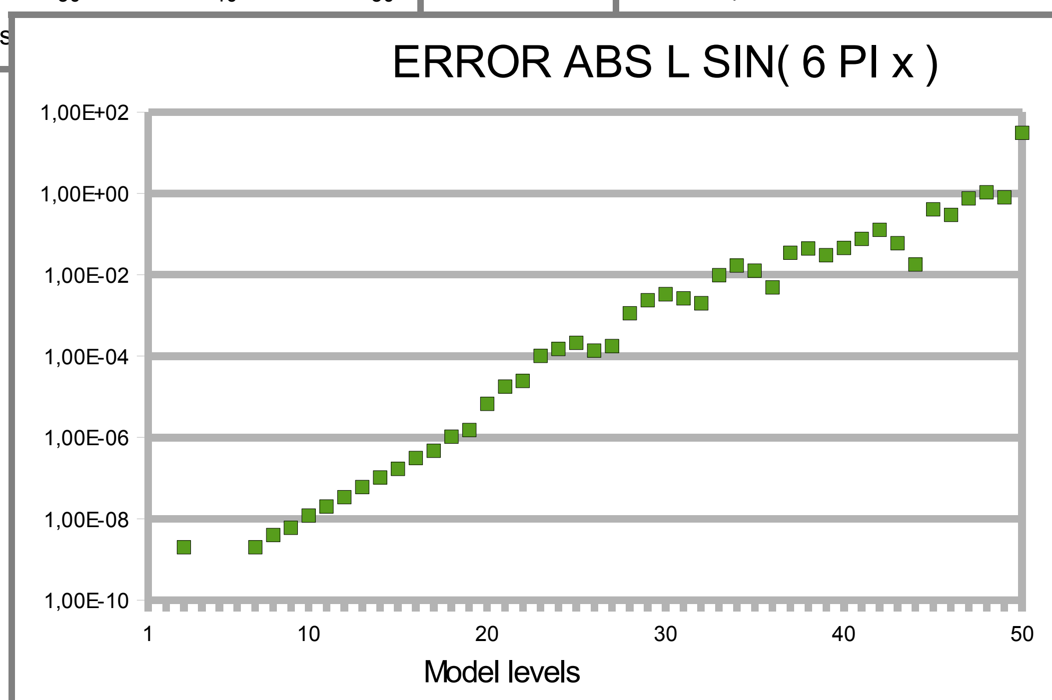
Test $\frac{\partial}{\partial \eta} X(\eta)$



Test $L^{*}X(\eta)=(\tilde{\partial}+1)\tilde{\partial}X(\eta)$







Vertical Laplacian and its eigenvalues

- Vertical Laplacian eigenvalues should be real and negative

$$(I-\delta t^2 c^{*2}(\Delta+\frac{L^*}{H^{*2}})-\delta t^4 N^{*2} c^{*2}\Delta)\chi=\hat{\chi}$$

- In practice VFE Laplacian eigenvalues depend on the number of levels and where the levels are placed
- When using B-splines some positive eigenvalues are present because $\eta^1, \eta^2, \ldots$ are eigenfuctions

$$L^*\eta^p=\eta\frac{\partial}{\partial\eta}(1+\eta\frac{\partial}{\partial\eta})\eta^p=p(1+p)\eta^p$$

- Some eigenvalues have imaginary parts
- But it is possible to modify the discrete vertical Laplacian to in order to get all the eigenvalues real and negative

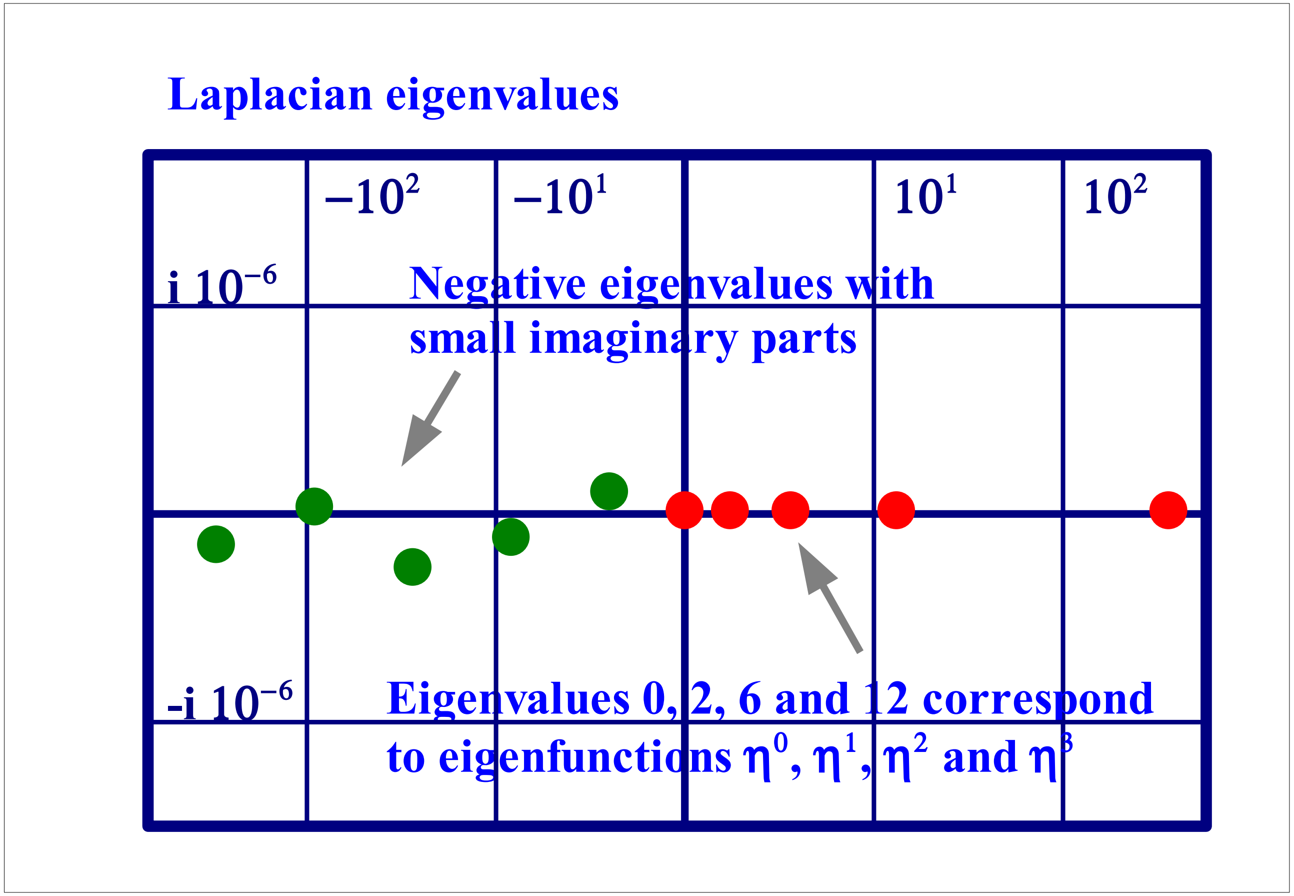
Modifying eigenvalues of linear operators

- Problem: given a $N \times N$ matrix $\boldsymbol{B}$ with eigenvalues $\{\beta_1, \ldots, \beta_N\}$ and given a new set of eigenvalues $\{\beta_1', \ldots, \beta_N'\}$ it is possible to find a modified matrix $\boldsymbol{B}'$ such that it differs from $\boldsymbol{B}$ only in one of its lines and such that its eigenvalues are $\{\beta_1', \ldots, \beta_N'\}$

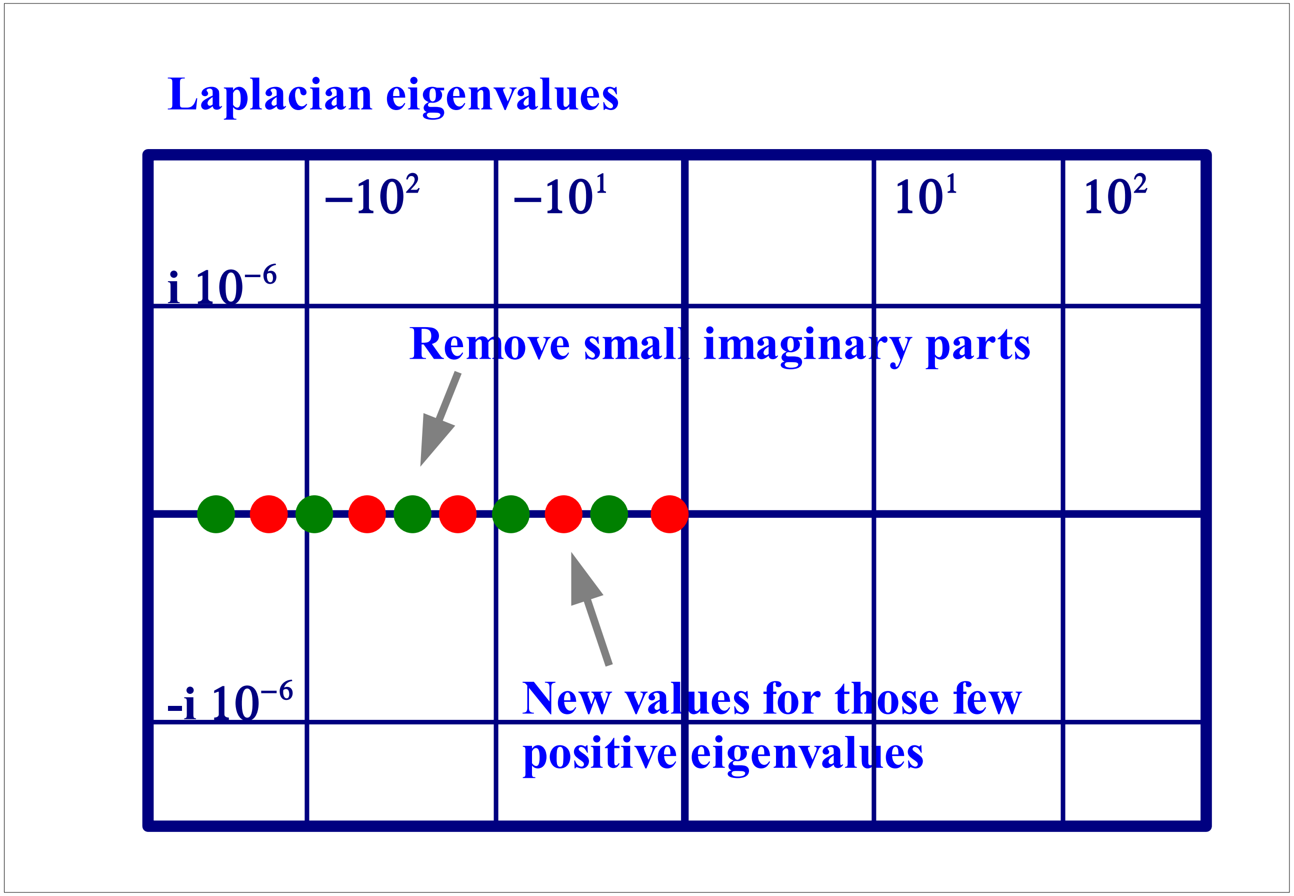
- Solution: for each new eigenvalue β_i' find a new eigenvector $\boldsymbol{v}_i' = \{1, v_{2i}', \ldots, v_{Ni}'\}^T$ such that $\sum_{k=1,N} B_{jk} v_{ki}' = \beta_i' v_{ji}'$ for $j = 1, \ldots, p-1, p+1, \ldots, N$. Then construct the new operator $\boldsymbol{B}' = \boldsymbol{v}' \cdot \boldsymbol{\beta}' \cdot \boldsymbol{v}'^{-1}$ where $\boldsymbol{\beta}'$ is a diagonal matrix with new eigenvalues and $\boldsymbol{v}'$ is a matrix with new eigenvectors. Operator $\boldsymbol{B}'$ is equal to $\boldsymbol{B}$ except line p and its eigenvalues and eigenvectors are $\{\beta_1', \ldots, \beta_N'\}$ and $\{\boldsymbol{v}_1', \ldots, \boldsymbol{v}_N'\}$ respectively

- This problem has a general solution. There are some restrictions on the new eigenvalues, for instance new eigenvectors must be linear independent. There is not a straightforward interpretation in the case $\boldsymbol{B}$ is a discretization of a continuous linear operator B

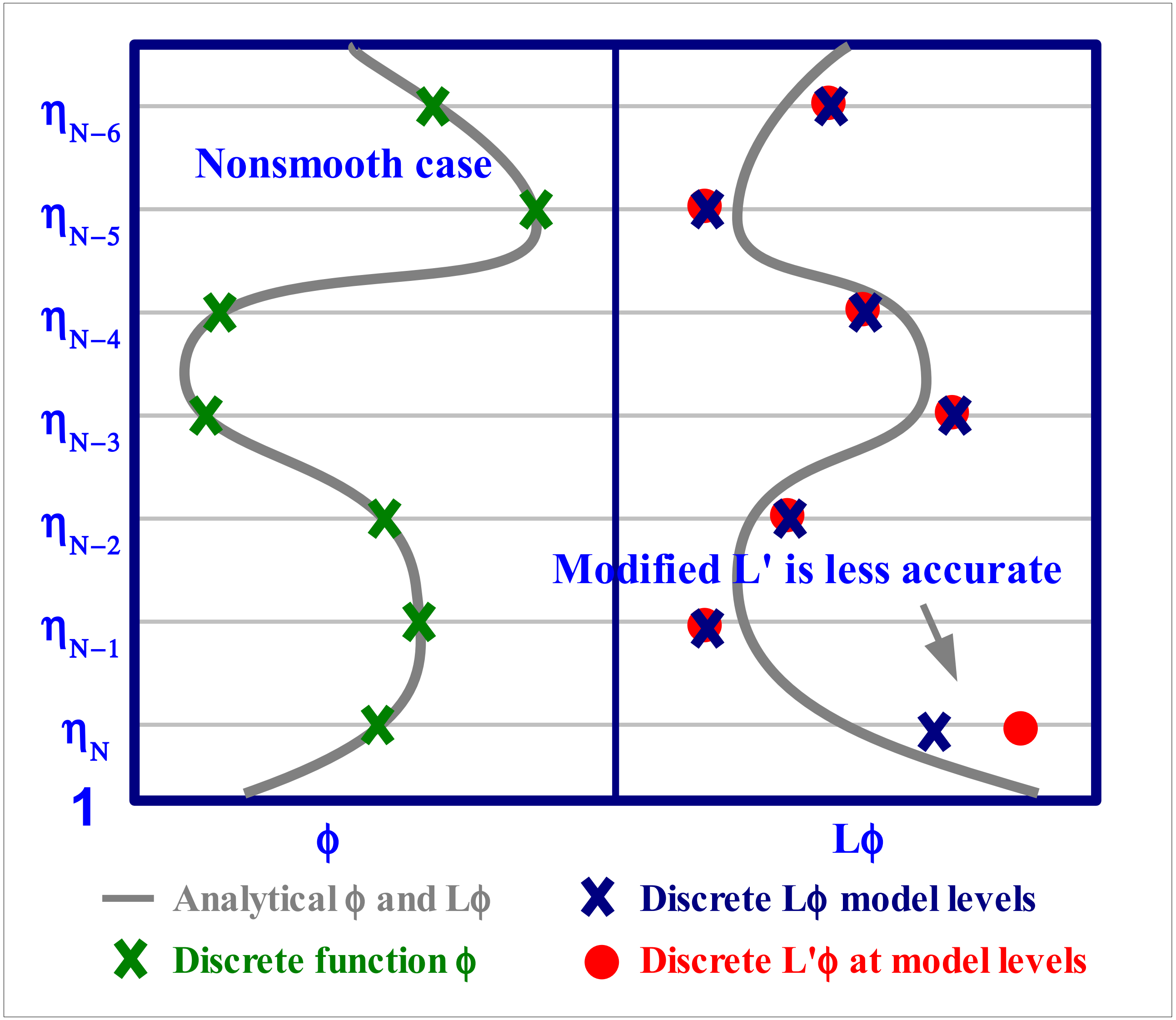
Modifying eigenvalues of vertical Laplacian



Modifying eigenvalues of vertical Laplacian



Modifying eigenvalues of vertical Laplacian



Accuracy of $\boldsymbol{L}'$

$\boldsymbol{L}'$ is equal to $\boldsymbol{L}$ at all levels except the lowest level

Accuracy of $\boldsymbol{L}'$ at the lowest level is worse than $\boldsymbol{L}$ and better than $\boldsymbol{L}_{\boldsymbol{FD}}$ finite differences scheme

Test of $\boldsymbol{L}$, $\boldsymbol{L}'$ and $\boldsymbol{L}_{\boldsymbol{FD}}$

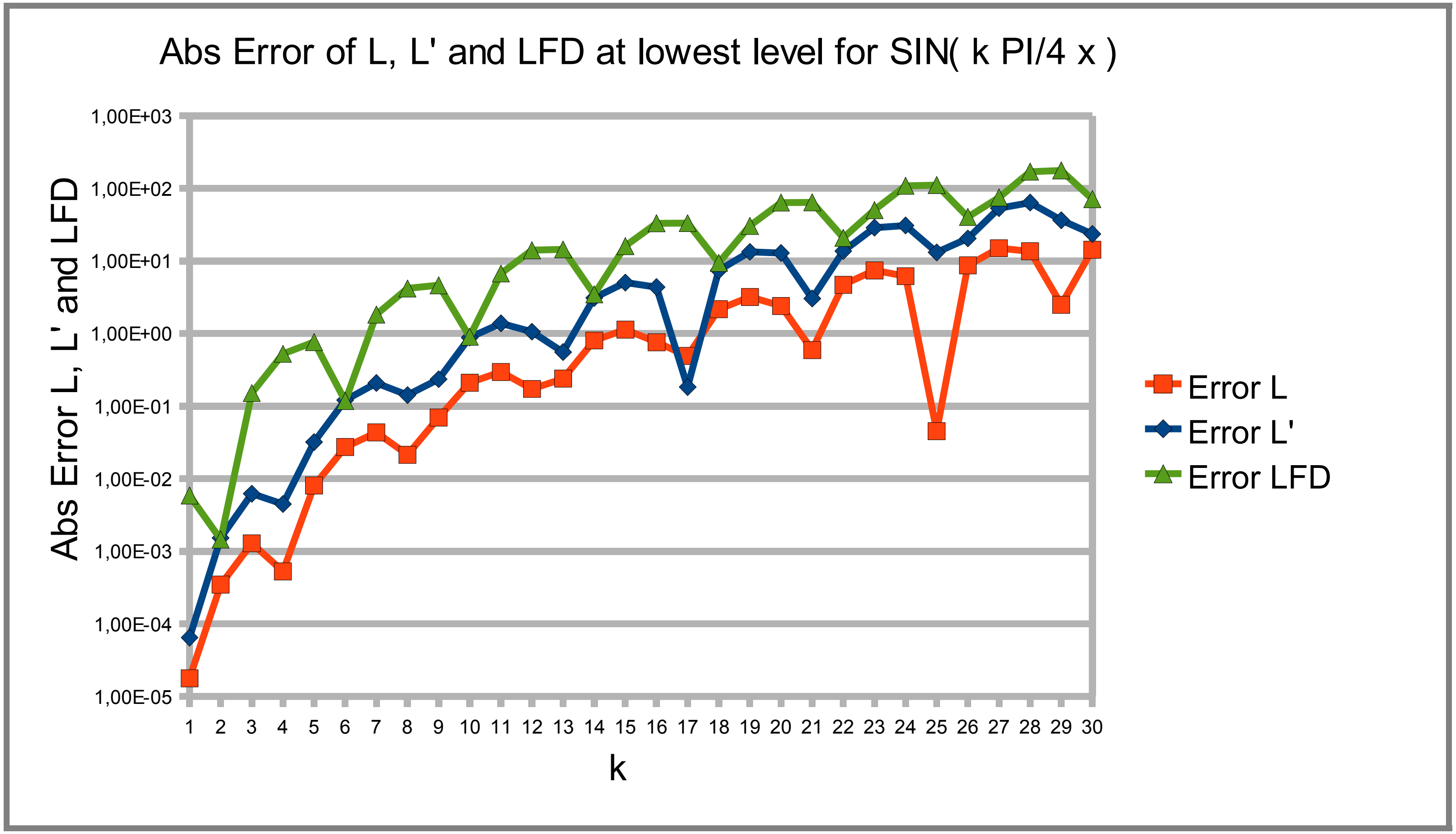
- function $\varphi_k(\eta)=\sin(k\pi/4\eta)$ for $k=1,\ldots,30$
- ECMWF 50 level model, test at level 50

$$\varepsilon(\boldsymbol{L},k)=|(\boldsymbol{L}\boldsymbol{\varphi}_k)_{50}-(L\varphi_k)(\eta_{50})|$$

$$\varepsilon(\boldsymbol{L}',k)=|(\boldsymbol{L}'\boldsymbol{\varphi}_k)_{50}-(L\varphi_k)(\eta_{50})|$$

$$\varepsilon(\boldsymbol{L}_{\boldsymbol{FD}},k)=|(\boldsymbol{L}_{\boldsymbol{FD}}\boldsymbol{\varphi}_k)_{50}-(L\varphi_k)(\eta_{50})|$$

Accuracy of *L'*



Conclusions

Linear systems and prognostic variables

- Different set of prognostic variables are proposed, not tested in a non linear model

VFE discretization of linear operators

- Similar to IFS VFE scheme, similar accuracy of integral operators
- Spline space and physical space have same dimension
- More straightforward method for higher order of accuracy
- Same scheme for integral and derivative

Modifying Vertical Laplacian

- VFE Laplacian has positive eigenvalues
- It is proposed a general method for modifying linear operators and eigenvalues
- Discontinuity in the accuracy of the modified operator can introduce noise

Future work

- Test operators with higher order splines, degree 4 and 5?
- Work in reducing accuracy discontinuity in the modified vertical Laplacian, optimization in the selection of new eigenvalues
- Implementation in IFS/HARMONIE of some of this results?

Some references

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Thank you for your attention

Gracias por vuestra atención