Runge-Kutta Time Integration and High-Order Spatial Discretization of Advection – a New Dynamical Core for the LM

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Two different Runge-Kutta schemes are implemented in the LM and combined with a forward-backward scheme for integrating high-frequency modes of the elastic equations in a time-splitting method. The first one is the 3rd-order Runge-Kutta scheme used by Wicker and Skamarock (2002) whereas the second one is a total variation diminishing (TVD) variant of 3rd-order (Shu and Osher 1988; Liu, Osher, and Chan 1994).

For horizontal advection upwind or centered-differences schemes of 3rd- to 6th-order can be used – the operators are formulated in advection form. The vertical advection is normally treated in an implicit way using a Crank-Nicolson scheme and centered-differences in space. As an alternative explicit schemes of 3rd-order or 4th-order can be used. Lower order schemes in time (2nd-order Runge-Kutta) and in space are also implemented as further options.
Most slow tendencies such as thermal/solar heating, parameterized convection, coriolis force and bouyancy are computed only once using values of the prognostic variables at time step n. These tendencies are fixed during the individual Runge-Kutta steps and contribute to the total slow-mode tendencies which are integrated in several small time steps together with the fast-mode tendencies in a time-splitting sense. In contradiction to this, the advection (as usual) and the vertical diffusion are computed in each of the Runge-Kutta steps. Especially the inclusion of the vertical mixing terms, which are also treated implicit, seems to stabilize the whole scheme.
In the following the procedure is described mathematically in a simplified form – the treatment of the physical forcings is omitted and the only operators clearly defined are the ones for advection.

Results of an advection test problem of a tracer in a deformational flow field (Durran 1999) are given in Figure 1. The number of time steps used for the stable integration of one deformation cycle is given in the caption for each of the different schemes. The initialized field was a cone with a maximum of 1.0 and a radius of 15 grid spacings.

To test the robustness of the scheme, the winter storm case "Lothar" (26 December 1999) was simulated with the LM. The maximum horizontal velocity during the simulation reaches 108 m/s. For this case the new scheme in the combination RK-3rd/UP-5th allows a time step of 72 s at a resolution of 7 km compared to a time step of 40 s of the operational leapfrog/CD-2nd scheme. Results are shown in Figure 2.

Problem to Solve:

$$\frac{\partial \phi}{\partial t} = L^{slow}(\phi) + L^{fast}(\phi)$$

Computation of the Slow Tendency:

Normal 3rd-order Runge-Kutta:

$$\begin{aligned}
\phi^{*}_{i,k} &= \phi^{n}_{i,k} - \frac{1}{3}\Delta t\; L^{h}_{i}(\phi^{n}) - \frac{1}{3}\Delta t\Big(\beta^{+} L^{v}_{k}(\phi^{*}) + \beta^{-} L^{v}_{k}(\phi^{n})\Big) \\
&= \phi^{0}_{i,k} + \frac{1}{3}\Delta t\; L^{slow}_{i,k}\Big|_{0}^{*}
\end{aligned}$$

$$\begin{aligned}
\phi^{**}_{i,k} &= \phi^{n}_{i,k} - \frac{1}{2}\Delta t\; L^{h}_{i}(\phi^{*}) - \frac{1}{2}\Delta t\Big(\beta^{+} L^{v}_{k}(\phi^{**}) + \beta^{-} L^{v}_{k}(\phi^{*})\Big) \\
&= \phi^{0}_{i,k} + \frac{1}{2}\Delta t\; L^{slow}_{i,k}\Big|_{0}^{**}
\end{aligned}$$

$$\begin{aligned}
\phi^{n+1}_{i,k} &= \phi^{n}_{i,k} - \Delta t\; L^{h}_{i}(\phi^{**}) - \Delta t\Big(\beta^{+} L^{v}_{k}(\phi^{n+1}) + \beta^{-} L^{v}_{k}(\phi^{**})\Big) \\
&= \phi^{0}_{i,k} + \Delta t\; L^{slow}_{i,k}\Big|_{0}^{n+1}
\end{aligned}$$

TVD-variant of 3rd-order Runge-Kutta:

$$\begin{aligned}
\phi^{*}_{i,k} &= \phi^{n}_{i,k} - \Delta t\; L^{h}_{i}(\phi^{n}) - \Delta t\Big(\beta^{+} L^{v}_{k}(\phi^{*}) + \beta^{-} L^{v}_{k}(\phi^{n})\Big) \\
&= \phi^{0}_{i,k} + \Delta t\; L^{slow}_{i,k}\Big|_{0}^{*}
\end{aligned}$$

$$\begin{aligned}
\phi^{**}_{i,k} &= \frac{3}{4}\phi^{n}_{i,k} + \frac{1}{4}\phi^{*}_{i,k} - \frac{1}{4}\Delta t\; L^{h}_{i}(\phi^{*}) - \frac{1}{4}\Delta t\Big(\beta^{+} L^{v}_{k}(\phi^{**}) + \beta^{-} L^{v}_{k}(\phi^{*})\Big) \\
&= \phi^{0}_{i,k} + \frac{1}{4}\Delta t\; L^{slow}_{i,k}\Big|_{0}^{**}
\end{aligned}$$

$$\begin{aligned}
\phi^{n+1}_{i,k} &= \frac{1}{3}\phi^{n}_{i,k} + \frac{2}{3}\phi^{**}_{i,k} - \frac{2}{3}\Delta t\; L^{h}_{i}(\phi^{**}) - \frac{2}{3}\Delta t\Big(\beta^{+} L^{v}_{k}(\phi^{n+1}) + \beta^{-} L^{v}_{k}(\phi^{**})\Big) \\
&= \phi^{0}_{i,k} + \frac{2}{3}\Delta t\; L^{slow}_{i,k}\Big|_{0}^{n+1}
\end{aligned}$$

Time-Splitting Method:

After each Runge-Kutta step the fast modes are integrated forward to the desired point in time using several small time steps $\Delta\tau$ – the slow tendency is fixed. The starting point of the integration $\phi^0_{i,k}$ depends on the choosen variant of the Runge-Kutta scheme – for the first variant it is always equal to $\phi^n_{i,k}$:

1. step:

$$\phi^{0+\Delta\tau}_{i,k} = \phi^0_{i,k} + \Delta\tau\; L^{fast}_{i,k}(\phi^0) + \Delta\tau\; L^{slow}_{i,k}\Big|_0^{\times}$$

remaining steps:

$$\phi^{\tau+\Delta\tau}_{i,k} = \phi^\tau_{i,k} + \Delta\tau\; L^{fast}_{i,k}(\phi^\tau) + \Delta\tau\; L^{slow}_{i,k}\Big|_0^{\times}$$

with $\times = *$, $**$ and $n+1$ in the individual Runge-Kutta steps.

Horizontal and Vertical Operators:

$$L^h_i(\phi)^{\,(4\mathrm{th})} = \frac{u_i}{12\Delta x}\Big[\phi_{i-2} - 8\big(\phi_{i-1} - \phi_{i+1}\big) - \phi_{i+2}\Big]$$

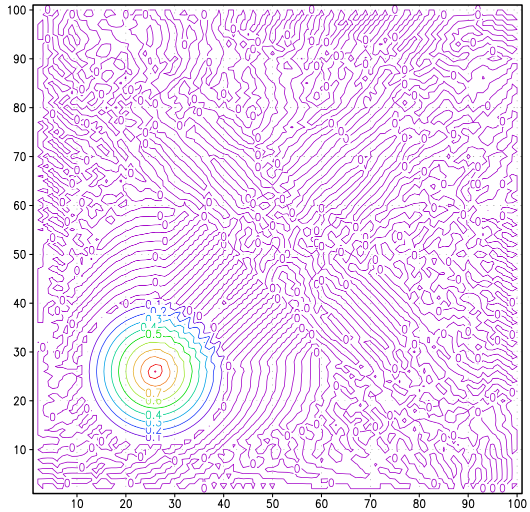
$$L^h_i(\phi)^{\,(3\mathrm{rd})} = L^h_i(\phi)^{\,(4\mathrm{th})} + \frac{|u_i|}{12\Delta x}\Big[\phi_{i-2} - 4\big(\phi_{i-1} + \phi_{i+1}\big) + 6\phi_i + \phi_{i+2}\Big]$$

$$L^h_i(\phi)^{\,(6\mathrm{th})} = \frac{u_i}{60\Delta x}\Big[-\phi_{i-3} + 9\big(\phi_{i-2} - \phi_{i+2}\big) - 45\big(\phi_{i-1} - \phi_{i+1}\big) + \phi_{i+3}\Big]$$

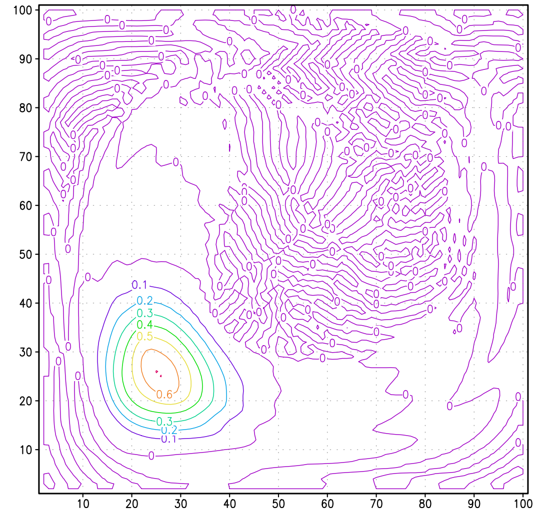
$$L^h_i(\phi)^{\,(5\mathrm{th})} = L^h_i(\phi)^{\,(6\mathrm{th})} + \frac{|u_i|}{60\Delta x}\Big[-\phi_{i-3} + 6\big(\phi_{i-2} + \phi_{i+2}\big) - 15\big(\phi_{i-1} + \phi_{i+1}\big) + 20\phi_i - \phi_{i+3}\Big]$$

$$L^v_k(\phi)^{\,(2\mathrm{nd})} = \frac{w_k}{2\Delta z}\big(\phi_{k+1} - \phi_{k-1}\big) + M^v_k(\phi)$$

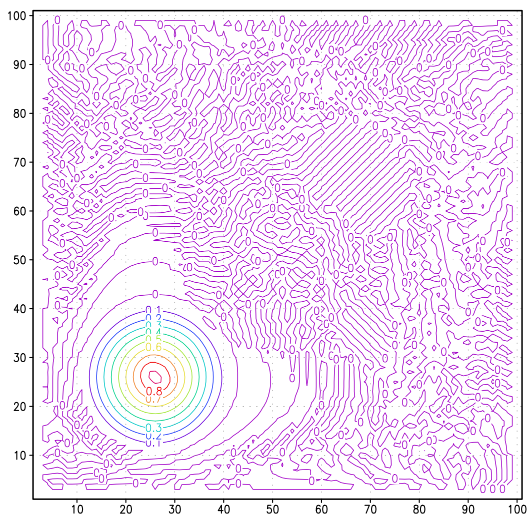
$M^v_k(\phi)$: vertical turbulent mixing term.



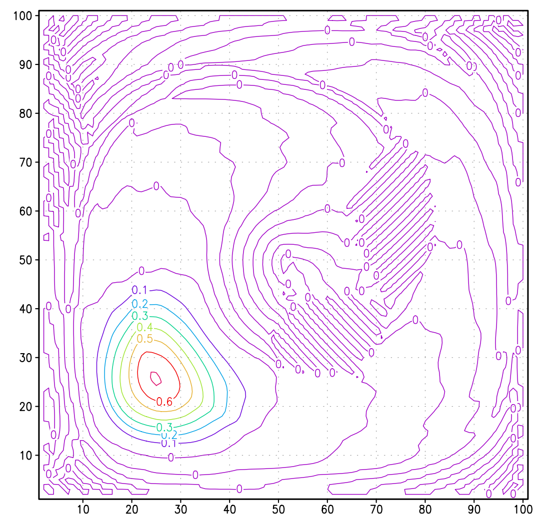
(a) RK-3rd / CD-4th – 670 time steps.



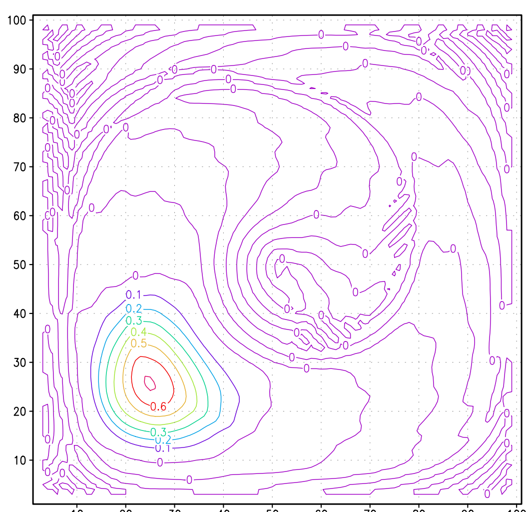
(b) RK-3rd / CD-4th with 4th-order artificial horizontal diffusion – 550 time steps.



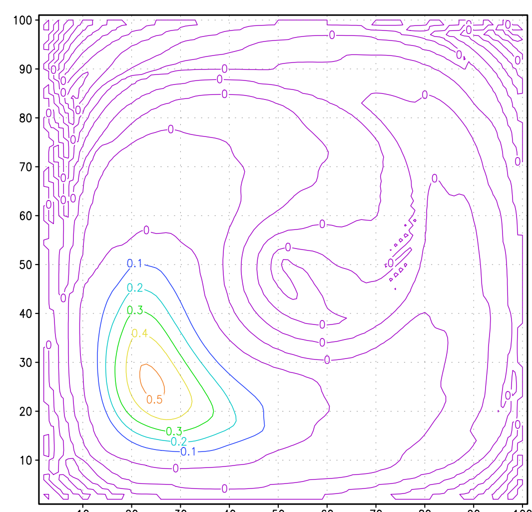
(c) TVD-RK-3rd / CD-4th – 450 time steps.



(d) RK-3rd / UP-5th – 380 time steps.

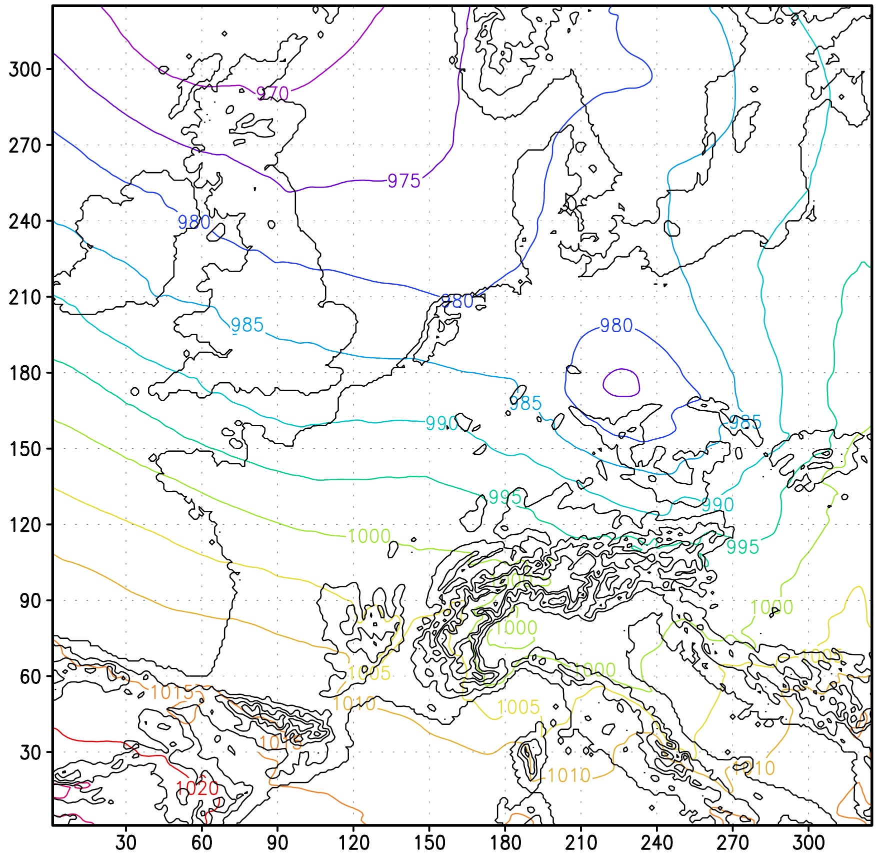


(e) TVD-RK-3rd / UP-5th – 380 time steps.

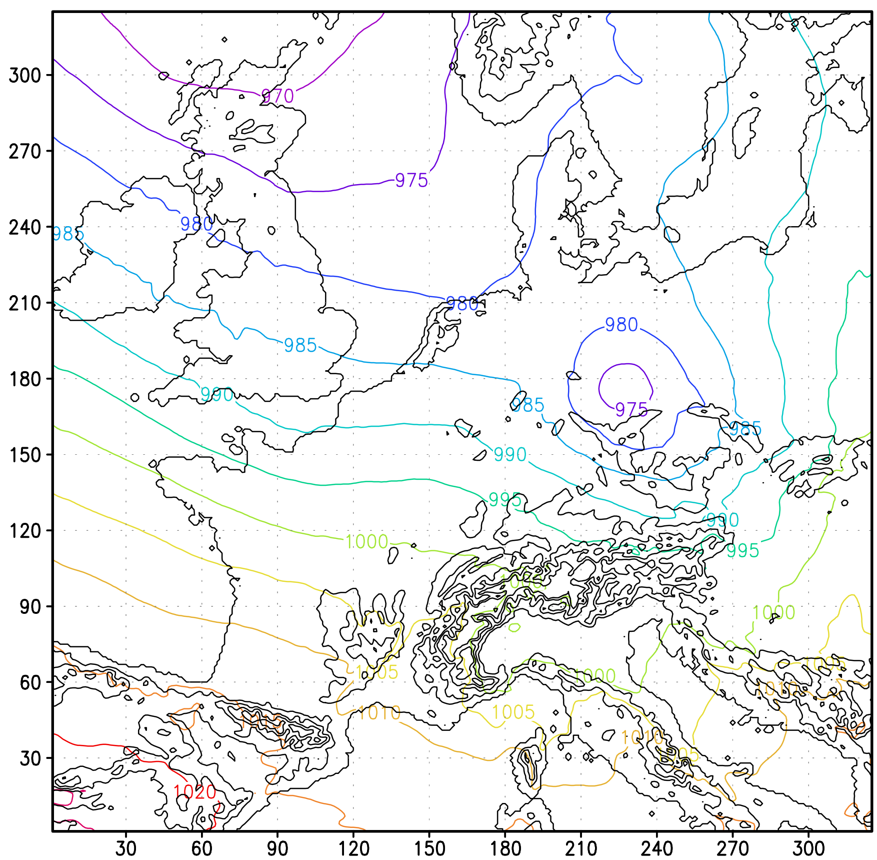


(f) TVD-RK-3rd / UP-3rd – 310 time steps.

Figure 1: *Advection of a tracer in a nondivergent deformational flow (Durran 1999). Results after 5 s simulation (one deformation cycle).*



(a) Leapfrog / CD-2nd – $\Delta t = 40$ s.



(b) TVD-RK-3rd / UP-5th – $\Delta t = 72$ s.

Figure 2: *Winter sorm "Lothar": mean sea level pressure in hPa – 26 December 1999, 16 UTC.*

References

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