Conservation of Total Energy and Total Momentum in Nonhydrostatic Numerical Models

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Background

- Conservation of fundamental properties (total mass, momentum, energy) has rarely been enforced in nonhydrostatic numerical models
- Reasons include:
  - simplicity
  - efficiency
  - short integration times (hours or days)

Motivation

- Conservation has recently become a primary design feature of some nonhydrostatic modeling systems (e.g., WRF Model in USA, NICAM in Japan)
- Reasons:
  - Transport and dispersion applications
  - Long-term integrations for climate studies
  - Some theoretical studies require greater precision (e.g., intensity of tropical cyclones)

Definition

- “conservation” is defined herein as both:

1. global conservation of a fundamental variable (such as total mass, total momentum, and total energy)
2. local conservation during application of a numerical algorithm … e.g., flux of mass out of a control volume = flux of mass into a neighboring control volume

Scope of this study

- Goal of this project has been to develop techniques that allow conservation of total mass, energy, and momentum for *compressible, split-explicit* nonhydrostatic models
  - Compressible: use un-approximated equations
  - Split-explicit: use a small timestep for acoustic modes, large timestep for other tendencies

A traditional approach:

Integrate equations for pressure (π),
velocity (u,w), and potential temperature (θ):

$$\frac{\partial u}{\partial t} + c_p\theta\frac{\partial \pi'}{\partial x} = -u\frac{\partial u}{\partial x} - w\frac{\partial u}{\partial z}$$

$$\frac{\partial w}{\partial t} + c_p\theta\frac{\partial \pi'}{\partial z} = -u\frac{\partial w}{\partial x} - w\frac{\partial w}{\partial z} + g\frac{\theta'}{\overline{\theta}}$$

$$\frac{\partial \pi'}{\partial t} + \pi\frac{R}{c_v}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = -u\frac{\partial \pi}{\partial x} - w\frac{\partial \pi}{\partial z}$$

$$\frac{\partial \theta'}{\partial t} = -u\frac{\partial \theta}{\partial x} - w\frac{\partial \theta}{\partial z}$$

$$\pi \equiv \left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}$$

Conservative equations:

Want to integrate equations for density (ρ), momentum ($U = \rho u, W = \rho w$), and entropy ($\Theta = \rho\theta$):

$$\frac{\partial U}{\partial t} + \frac{\partial p'}{\partial x} = -\frac{\partial U u}{\partial x} - \frac{\partial W u}{\partial z}$$

$$\frac{\partial W}{\partial t} + \frac{\partial p'}{\partial z} + g\rho' = -\frac{\partial U w}{\partial x} - \frac{\partial W w}{\partial z}$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial W}{\partial z} = 0$$

$$\frac{\partial \Theta}{\partial t} + \frac{\partial U\theta}{\partial x} + \frac{\partial W\theta}{\partial z} = 0$$

$$p = p_0 \left(\frac{R\Theta}{p_0}\right)^{\frac{c_p}{c_v}}$$

e.g., Klemp et al. (2007, MWR)

Solution part 1:

recast momentum variables in terms of perturbations from a recent time t:

$$U = U^t + U''$$
$$W = W^t + W''$$

$$\frac{\partial U''}{\partial t} + \frac{\partial p'}{\partial x} = -\frac{\partial U^t u}{\partial x} - \frac{\partial W^t u}{\partial z}$$
$$\frac{\partial W''}{\partial t} + \frac{\partial p'}{\partial z} + g\rho' = -\frac{\partial U^t w}{\partial x} - \frac{\partial W^t w}{\partial z}$$
$$\frac{\partial \rho'}{\partial t} + \frac{\partial U''}{\partial x} + \frac{\partial W''}{\partial z} = -\frac{\partial U^t}{\partial x} - \frac{\partial W^t}{\partial z}$$
$$\frac{\partial \Theta}{\partial t} + \frac{\partial U''\theta}{\partial x} + \frac{\partial W''\theta}{\partial z} = -\frac{\partial U^t\theta}{\partial x} - \frac{\partial W^t\theta}{\partial z}$$

Klemp et al. (2007, MWR)

Solution part 2:

recast pressure gradients in terms of Θ
(using ideal gas law)

$$p = p_0 \left(\frac{R\Theta}{p_0} \right)^{\frac{c_p}{c_v}}$$

$$\frac{\partial U''}{\partial t} + \frac{c_p}{c_v} R\pi \frac{\partial \Theta'}{\partial x} = -\frac{\partial U^t u}{\partial x} - \frac{\partial W^t u}{\partial z}$$

$$\frac{\partial W''}{\partial t} + \frac{c_p}{c_v} R\pi \frac{\partial \Theta'}{\partial z} + g\left(\rho' - \overline{\rho}\frac{\pi'}{\overline{\pi}} \right) = -\frac{\partial U^t w}{\partial x} - \frac{\partial W^t w}{\partial z}$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial U''}{\partial x} + \frac{\partial W''}{\partial z} = -\frac{\partial U^t}{\partial x} - \frac{\partial W^t}{\partial z}$$

$$\frac{\partial \Theta'}{\partial t} + \frac{\partial U''\theta}{\partial x} + \frac{\partial W''\theta}{\partial z} = -\frac{\partial U^t\theta}{\partial x} - \frac{\partial W^t\theta}{\partial z}$$

$\rightarrow$ In this system, total mass is conserved (locally and globally)

Unresolved issues:

- No guarantee of momentum conservation, owing to form of pressure-gradient terms

- No guarantee of total energy conservation
  - In fact, an approximation is typically made wherein dissipative heating is neglected
  - Dissipative heating is know to play an important role at high wind speeds (tropical cyclones) and long-term integrations (seasonal time scales)

Total Energy (E_t)

A different approach: use total energy, E_t, as a predictive variable

$$\underbrace{E_t}_{\text{total}} = \underbrace{\rho c_v T}_{\text{internal } (e)} + \underbrace{\rho g z}_{\text{potential } (\Phi)} + \underbrace{\rho \frac{1}{2}\left(u^2 + w^2\right)}_{\text{kinetic } (K)}$$

Note, from ideal gas law:

$$p = \rho R T = \frac{R}{c_v} e = \frac{R}{c_v}\left(E_t - \Phi - K\right)$$

Calculate pressure gradient in terms of E_t:

$$\frac{\partial p}{\partial x_i} = \frac{\partial}{\partial x_i}\left(\frac{R}{c_v}\left(E_t - \Phi - K\right)\right)$$

Integrate a governing equation for E_t:

$$\frac{\partial E_t}{\partial t} = -\frac{\partial U e^*}{\partial x} - \frac{\partial W e^*}{\partial z}$$

wherein

$$e^* \equiv \frac{E}{\rho} + p = \frac{E}{\rho} + \frac{R}{c_v}\left(E - \Phi - K\right)$$

New solution procedure:
use same techniques as before

$$\frac{\partial U''}{\partial t} + \frac{\partial}{\partial x}\left(\frac{R}{c_v}\left(E_t' - \rho' g z\right)\right) = -\frac{\partial U^t u}{\partial x} - \frac{\partial W^t u}{\partial z} + \frac{\partial}{\partial x}\left(\frac{R}{c_v}K\right)$$

$$\frac{\partial W''}{\partial t} + \frac{\partial}{\partial z}\left(\frac{R}{c_v}\left(E_t' - \rho' g z\right)\right) + g\rho' = -\frac{\partial U^t w}{\partial x} - \frac{\partial W^t w}{\partial z} + \frac{\partial}{\partial z}\left(\frac{R}{c_v}K\right)$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial U''}{\partial x} + \frac{\partial W''}{\partial z} = -\frac{\partial U^t}{\partial x} - \frac{\partial W^t}{\partial z}$$

$$\frac{\partial E_t'}{\partial t} + \frac{\partial U'' e^*}{\partial x} + \frac{\partial W'' e^*}{\partial z} = -\frac{\partial U^t e^*}{\partial x} - \frac{\partial W^t e^*}{\partial z}$$

Advantages:

- All terms are in flux form (except buoyancy) $\rightarrow$ conservation
- All variables on left side are either held fixed (e*) or are integrated on the small steps $\rightarrow$ efficiency and accuracy

Tests

- Developed a code that can integrate all three equation sets:
  - Non-conserving (u, w, π, θ)
  - Mass-conserving (U, W, ρ, Θ)
  - Mass,Momentum,Energy-conserving (U, W, ρ, E_t)
- Same techniques as WRF Model (ARW):
  - 3rd-order Runge-Kutta
  - 5th-order advection operators
  - (Cartesian height coordinate is different from ARW)

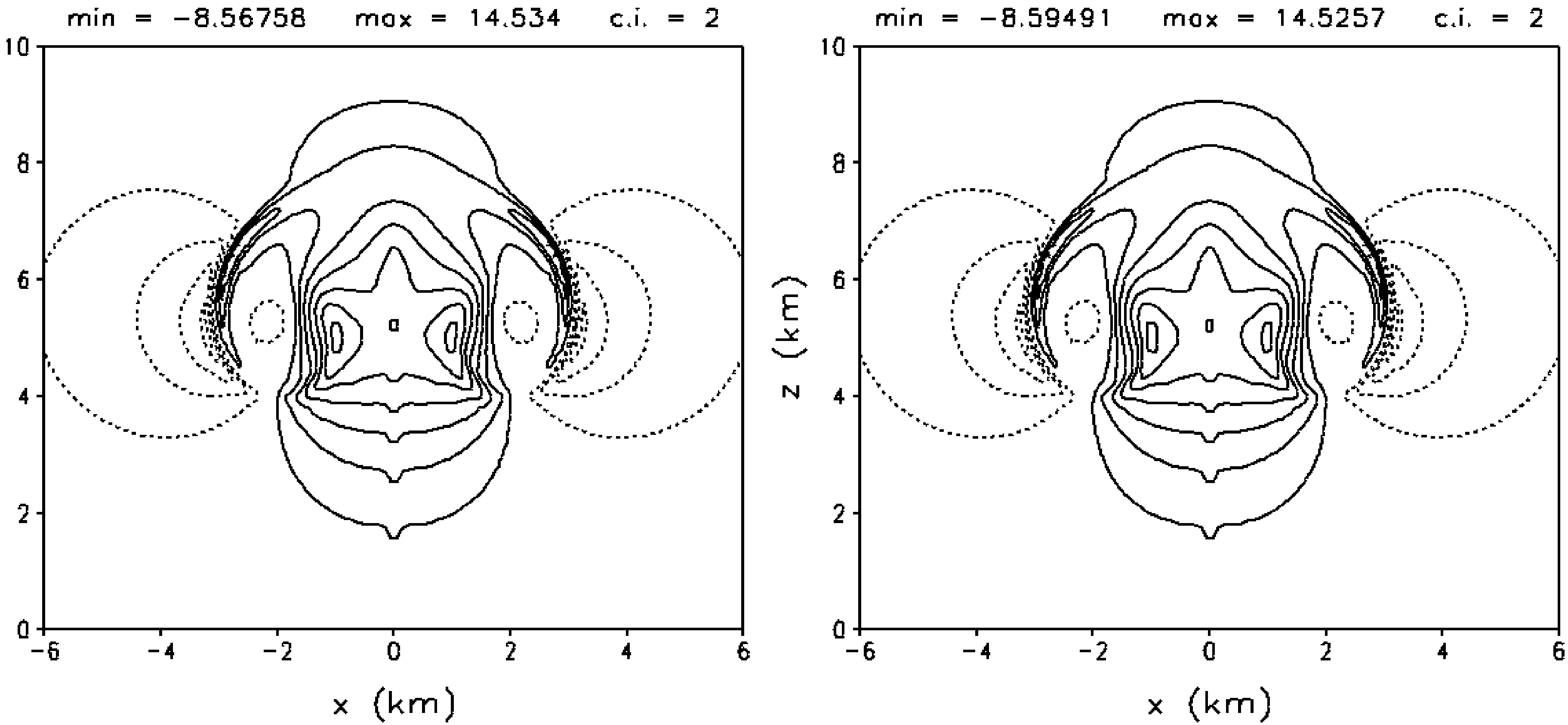
A simple test:

- Warm bubble ("moist benchmark") case used by Bryan and Fritsch (2002, MWR)
- No analytic solution, but:
  - well resolved (does not collapse to grid-scale)
  - well-known solution (produced by many models)
  - useful for testing dry and moist equations
- Details:
  - 2D, $\Delta x = \Delta z = 100$ m
  - Statically neutral initial state with warm bubble
  - Integrate for 1000 s

w (m/s) at t = 1000 s

Non-conserving

Mass-conserving

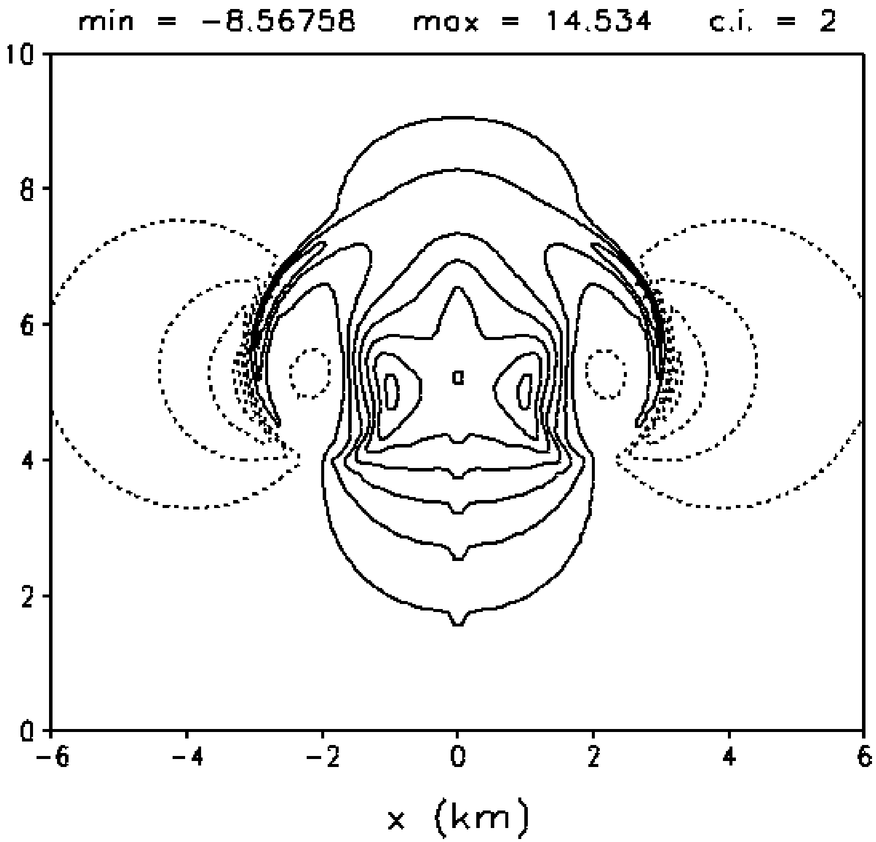


run time: 79 s

run time: 89 s

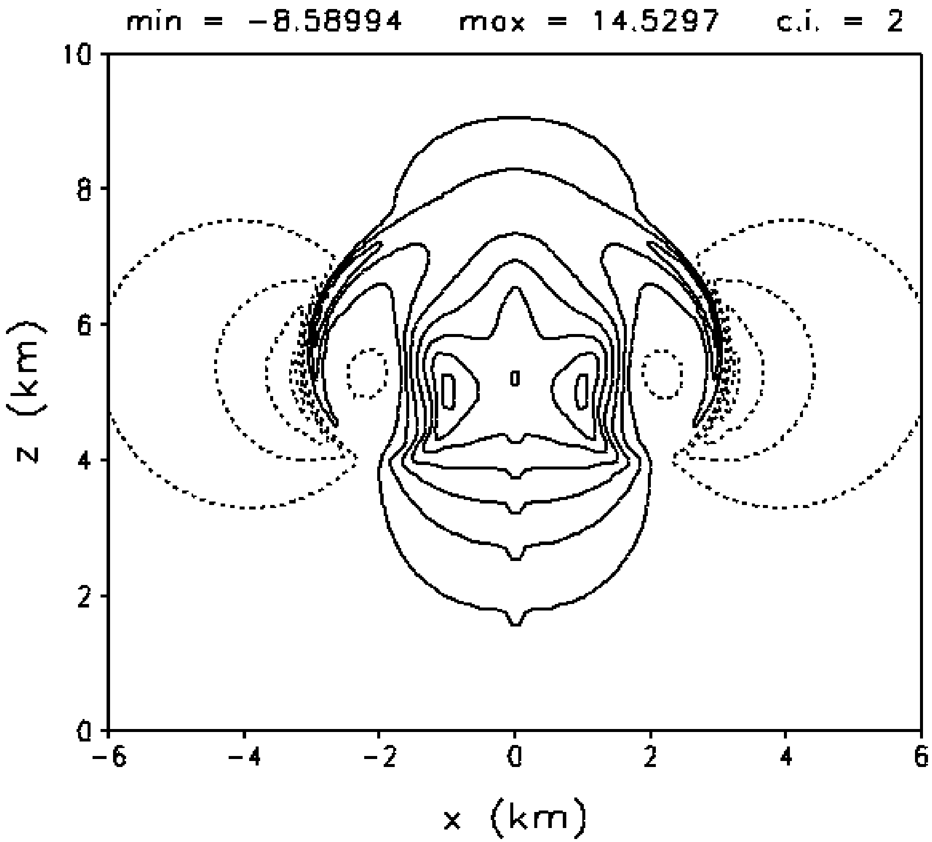
w (m/s) at t = 1000 s

Non-conserving



run time: 79 s

Mass,Mo,Ene-conserving

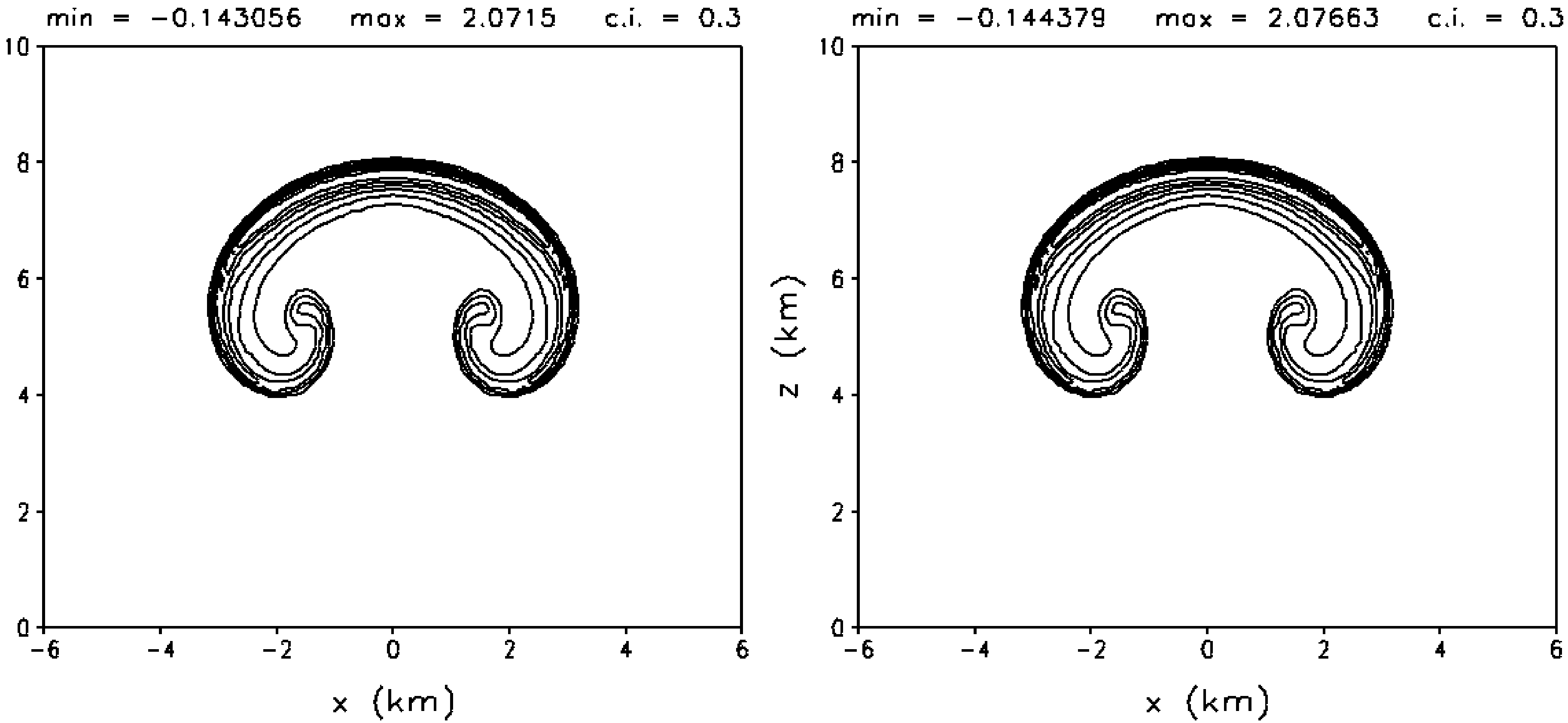


run time: 82 s

θ′ (K) at t = 1000 s

Non-conserving

Mass-conserving

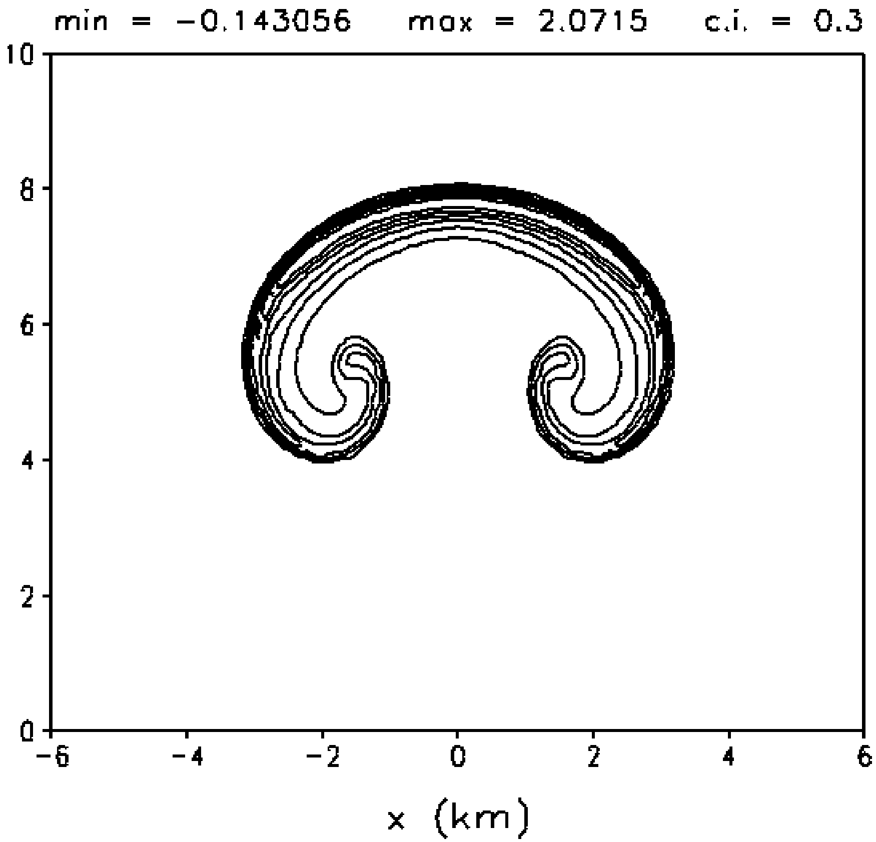


run time: 79 s

run time: 89 s

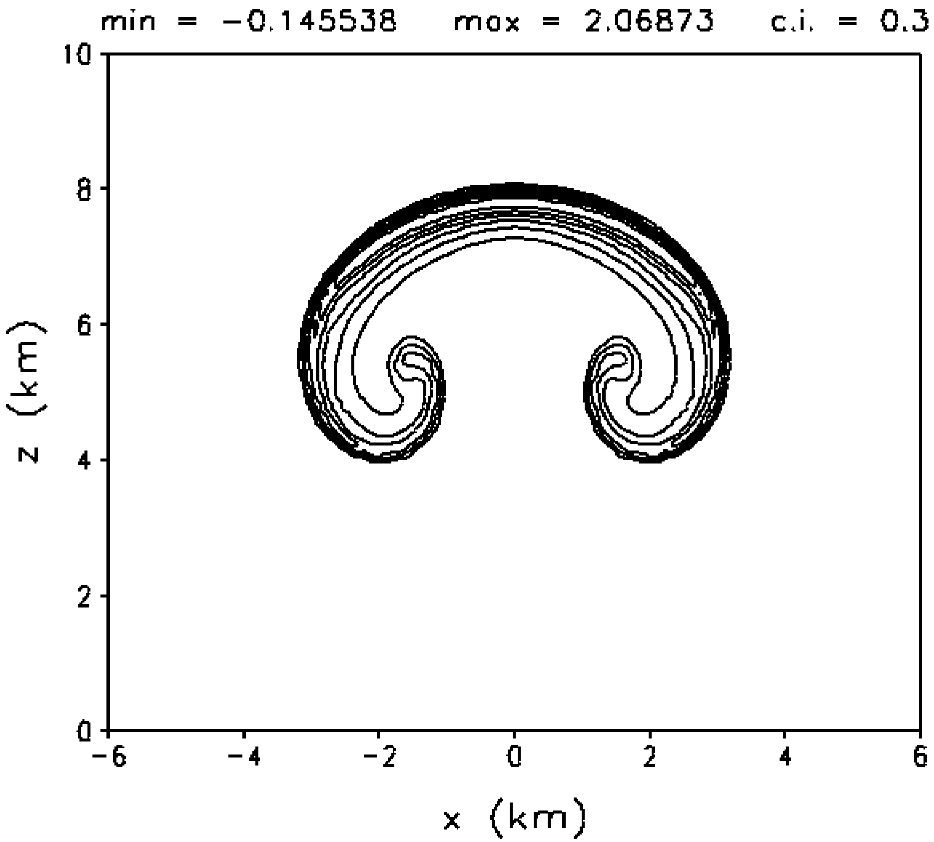
θ′ (K) at t = 1000 s

Non-conserving



run time: 79 s

Mass,Mo,Ene-conserving



run time: 82 s

Efficiency of dry bubble tests

- Run times:
  - Non-conserving: 79 s
    (fewer terms on small steps)
  - Mass-conserving: 89 s
    (more terms on small steps)
    (calculation of π is expensive)
  - Mass,Mo,Ene-conserving: 82 s
    (more terms on small steps)

Setup for moist comparison simulations:

$$\textbf{dry } (q_v = q_c = 0):$$

$$N^2 \equiv \frac{g}{\theta}\frac{d\theta}{dz} = 0$$

$$\textbf{moist, subsaturated } (q_v > 0, q_c = 0):$$

$$N_m^2 \equiv \frac{g}{\theta_v}\frac{d\theta_v}{dz} = 0$$

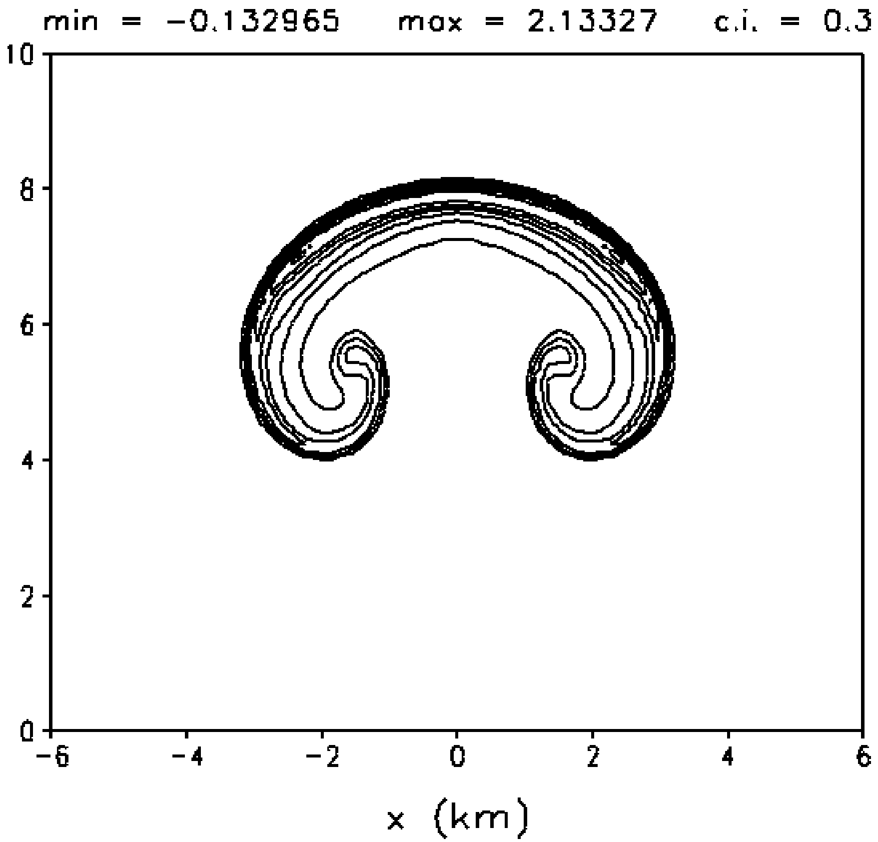
$$\textbf{moist, saturated } (q_v > 0, q_c > 0):$$

$$N_m^2 \equiv \frac{g}{T}\left(\frac{\partial T}{\partial z} + \Gamma_m\right)\left(1 + \frac{T}{R_d/R_v + q_s}\frac{\partial q_s}{\partial T}\right) - \frac{g}{1+q_t}\frac{\partial q_t}{\partial z} = 0$$

$$\text{wherein: } \Gamma_m \equiv g\,(1+q_t)\left(\frac{1 + Lq_s/R_dT}{c_{pm} + L\partial q_s/\partial T}\right)$$

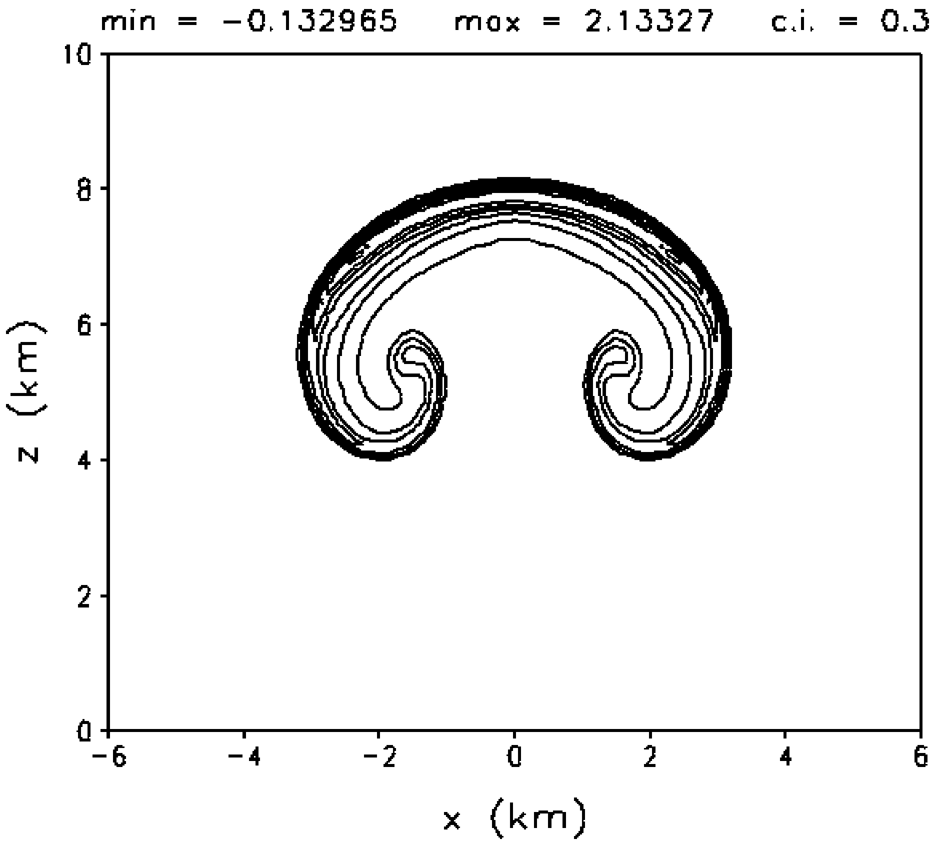
Moist, subsaturated case: θ_v' (K) at t = 1000 s

Non-conserving



run time: 105 s

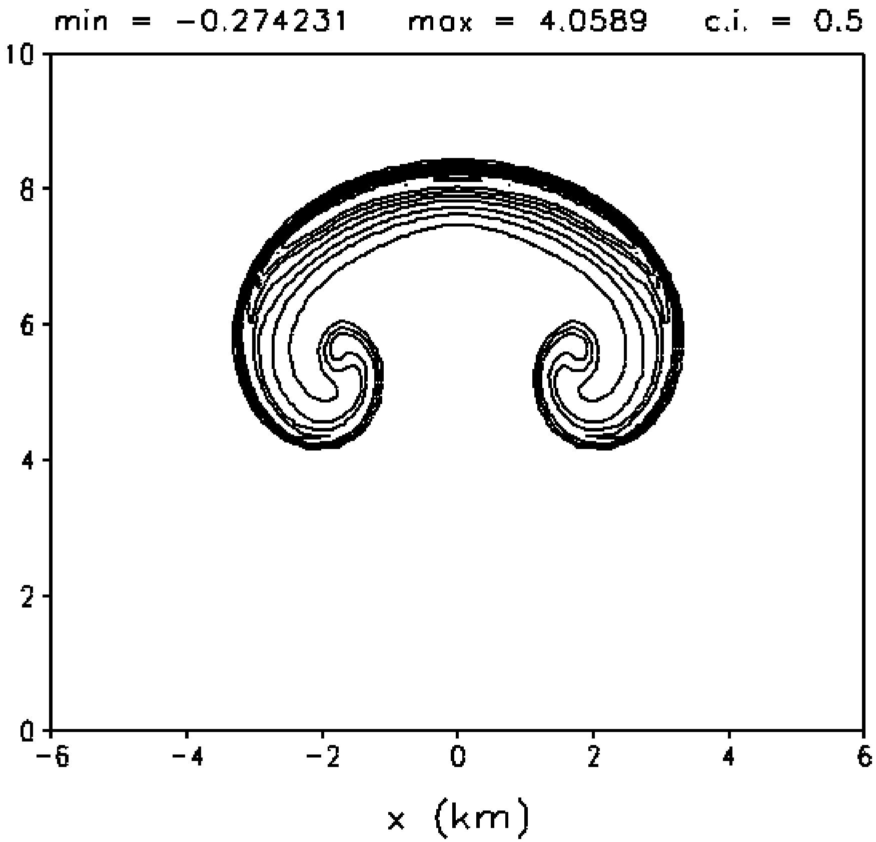
Mass,Mo,Ene-conserving



run time: 118 s

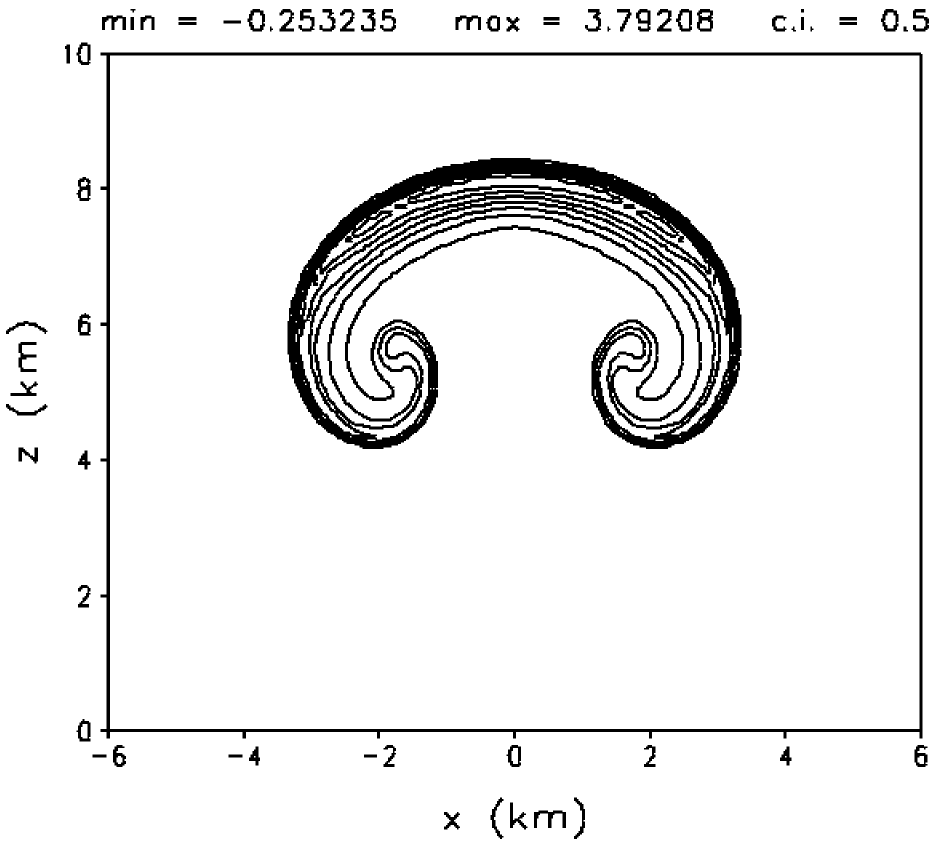
Moist, saturated case: θ_e' (K) at t = 1000 s

Non-conserving



run time: 111 s

Mass,Mo,Ene-conserving



run time: 121 s

Summary

- It is possible to formulate a nonhydrostatic solver that conserves (locally and globally) total mass, momentum, and energy
- All tendencies on small timesteps are calculated using the model's predictive variables: U,W,ρ,E_t … similar in design to traditional u,v,π,θ solvers
- Our prototype solver is competitive (run-time and RAM) with a traditional solver for both dry and moist flows