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An Inherently Mass-Conserving SISL Discretisation of the Nonhydrostatic Vertical Slice Equations

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Overview

- Background
- Coupling conservation to momentum equations
- Governing equations
- Discretisation and iterative solver
- Results
- Conclusions

Background

- Mass conservation is important for climate simulations but is difficult for SL schemes.
- Coupling of a SL conservative scheme to the momentum equations has been shown for the SWE's, (Zerroukat *et al.* 2009).
- This is extended to a vertical slice model with orography.
- Constant reference height of the SWE becomes a vertically varying reference profile.

Coupling the continuity equation to the momentum equations

- Standard (Lagrangian) continuity equation:

$$\frac{D\rho}{Dt} + \rho\nabla \cdot \mathbf{u} = 0$$

- Rewrite (more fundamentally) in conservative form:

$$\frac{D}{Dt}\left(\int_{\delta\mathcal{A}} \rho d\mathcal{A}\right) = 0$$

- $\mathcal{A} =$ vertical area element (control volume in 3D).
- Now apply SL scheme but key to stability is to couple directly to momentum equations

Coupling the continuity equation to the momentum equations

- Define a reference density ρ^{ref} and rewrite:

$$\frac{D}{Dt}\left[\int_{\delta\mathcal{A}} \left(\rho - \rho^{\mathrm{ref}}\right) d\mathcal{A}\right] + \int_{\delta\mathcal{A}} \nabla \cdot \left(\rho^{\mathrm{ref}} \mathbf{u}\right) d\mathcal{A} = 0$$

- Introduction of the reference profile allows a divergence term to be pulled off, coupling the momentum and continuity equations

Coupling the continuity equation to the momentum equations

- Discretising along a trajectory and taking arrival volume $\equiv$ control volume surrounding $\rho \equiv \Delta\mathcal{A}$:

$$\left[\left(\rho-\rho^{\mathrm{ref}}\right)+\alpha\Delta t\nabla\cdot\left(\rho^{\mathrm{ref}}\mathbf{u}\right)\right]_{A}^{n+1}\equiv$$

$$\frac{1}{\Delta\mathcal{A}}\left\{\int_{\delta\mathcal{A}}\left[\left(\rho-\rho^{\mathrm{ref}}\right)-\beta\Delta t\nabla\cdot\left(\rho^{\mathrm{ref}}\mathbf{u}\right)\right]d\mathcal{A}\right\}_{D}^{n},$$

- Conservation is assured if RHS integration is conservative, (here the SLICE algorithm is used).

Equations in height coordinates z

$$\frac{Du}{Dt} + c_p\theta\frac{\partial\pi}{\partial x} - fv + Fw = 0, \tag{1}$$

$$\frac{Dv}{Dt} + fu = 0, \tag{2}$$

$$\delta_V\frac{Dw}{Dt} + c_p\theta\frac{\partial\pi}{\partial z} + g - Fu = 0, \tag{3}$$

$$\frac{D}{Dt}\left(\int_{\delta\mathcal{A}}\rho d\mathcal{A}\right) = 0, \tag{4}$$

$$\frac{D\theta}{Dt} = 0, \tag{5}$$

$$\pi^{\frac{1-\kappa}{\kappa}} = \frac{R}{p_0}\rho\theta, \tag{6}$$

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... in terrain following coordinates $\eta(x, z)$

$$\frac{Du}{Dt} = -\frac{c_p\theta}{\partial z/\partial\eta}\left[\frac{\partial}{\partial x}\left(\pi\frac{\partial z}{\partial\eta}\right) - \frac{\partial}{\partial\eta}\left(\pi\frac{\partial z}{\partial x}\right)\right] + fv - Fw, \tag{7}$$

$$\frac{Dv}{Dt} = -fu, \tag{8}$$

$$\delta_V\frac{Dw}{Dt} = -\frac{c_p\theta}{\partial z/\partial\eta}\frac{\partial\pi}{\partial\eta} - g + Fu, \tag{9}$$

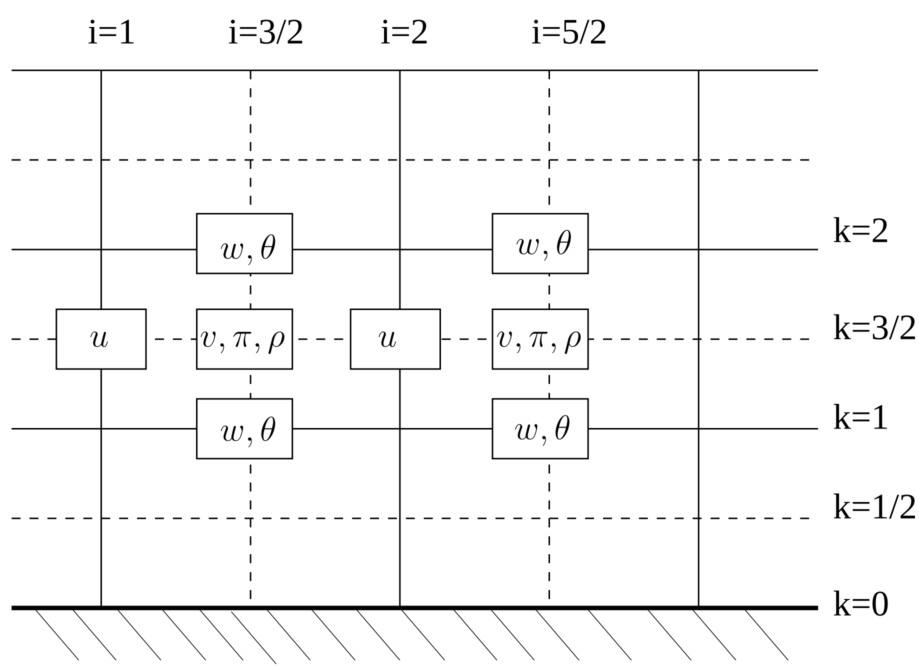
$$\frac{D}{Dt}\left[\int_{\delta\mathcal{A}}\left(\rho - \rho^{\mathrm{ref}}\right)d\mathcal{A}\right] + \int_{\delta\mathcal{A}}\nabla\cdot\left(\rho^{\mathrm{ref}}\mathbf{u}\right)d\mathcal{A} = 0, \tag{10}$$

$$\frac{D}{Dt}\left(\theta - \theta^{\mathrm{ref}}\right) + u\frac{\partial\theta^{\mathrm{ref}}}{\partial x} + \dot{\eta}\frac{\partial\theta^{\mathrm{ref}}}{\partial\eta} = 0, \tag{11}$$

$$\pi^{\frac{1-\kappa}{\kappa}} = \frac{R}{p_0}\rho\theta, \tag{12}$$

Discretisation

- Equations are discretised in time using a two-time-level SISL scheme.
- Staggered Arakawa-C grid is used in the horizontal.
- Charney-Phillips grid is used in the vertical.
- Cubic-Lagrange polynomials used to interpolate fields to departure points.



Model Features

- Coriolis terms are discretised following Thuburn and Staniforth (2004).
- Near boundary correction to departure point computations (Wood *et al.* 2009) is applied.
- A semi-Lagrangian discretisation of the relationship between $\dot{\eta}$ and w is used (following Girard *et al.* 2005).
  - Eulerian: $\dot{\eta} = \left(\frac{\partial z}{\partial \eta}\right)^{-1} \left(w - u\frac{\partial z}{\partial x}\right)$.
  - Lagrangian: $\dot{\eta} = \left(\frac{\partial z}{\partial \eta}\right)^{-1} \left(w - \frac{Dz}{Dt}\Big|_{\eta}\right)$.
- A variant of the absorbing layer (Klemp *et al.* 2008) is used to avoid spurious reflections from the domain lid.
- A multiply nested iterative approach (Zerroukat *et al.* 2009) is used to avoid any three-time-level extrapolations.

Iterative algorithm

Do time-step loop:

- given $(\dot{\eta}, \theta, w, u, v, \rho, \pi)^n$ at level n

Do outer-loop iteration:

- compute (x_D^n, z_D^n) using $(u, w)^n$ and latest estimate for $(u, w)^{n+1}$
- evaluate R_D^n for required fields R

Do inner-loop iteration:

- evaluate Coriolis and nonlinear terms
- solve Helmholtz problem for π^{n+1}
- update $(\dot{\eta}, \theta, w, u, v, \rho)^{n+1}$

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Computational examples

- Results tested on the standard test set:
  - Gravity waves.
  - Density Current.
  - Hydrostatic and nonhydrostatic mountain waves. [not shown]
  - Schär hill test.
- Linear $\eta - z$ transformation is used:

$$\eta = \frac{z - z_S(x)}{z_T - z_S(x)}$$

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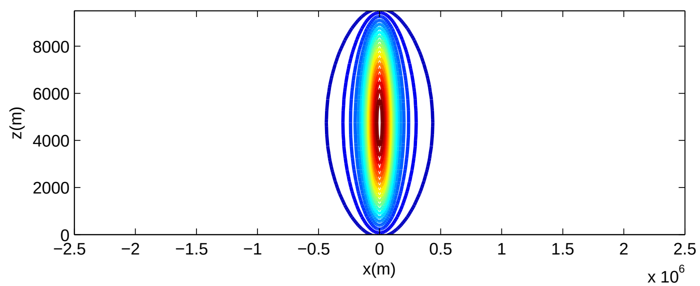
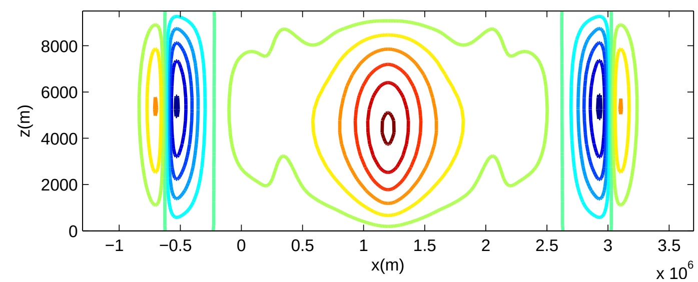
Gravity Waves

- Hydrostatic inertia gravity waves

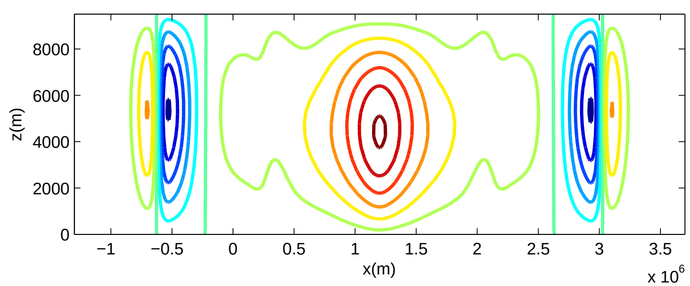
$t = 0$



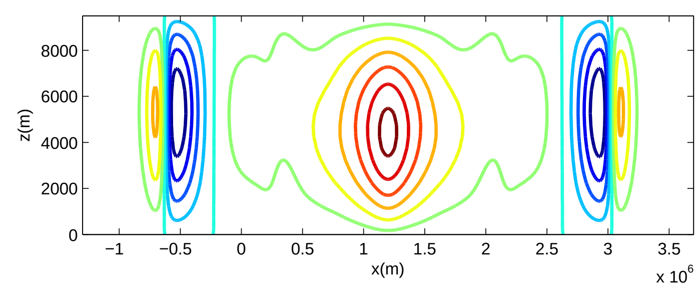
Mass-conserving $t = 1000$ mins



SISL $t = 1000$ mins,



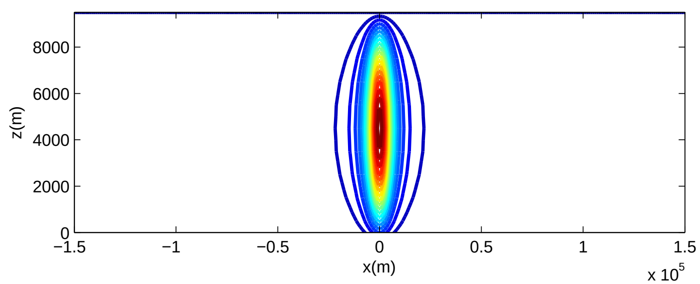
Mass-conserving, $\delta_V = 0$, $t = 1000$ mins



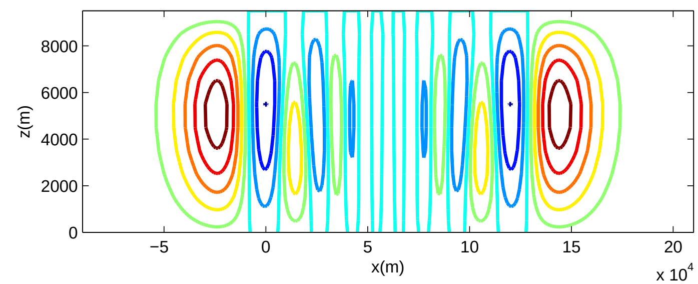
Gravity Waves

- Nonhydrostatic gravity waves

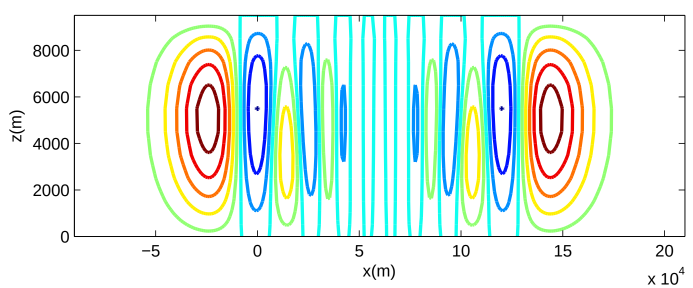
$t = 0$



Mass-conserving $t = 1000$ mins

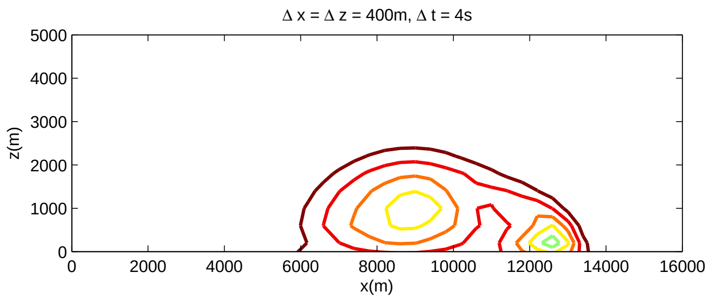


SISL $t = 1000$ mins

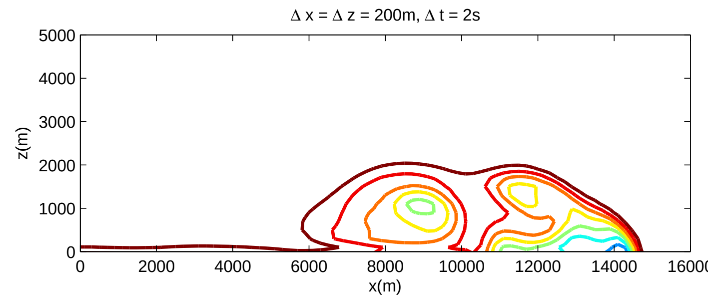


Density Current

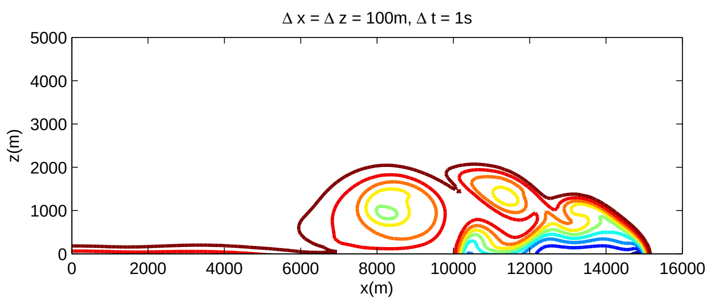
$$\Delta x = 400m,$$



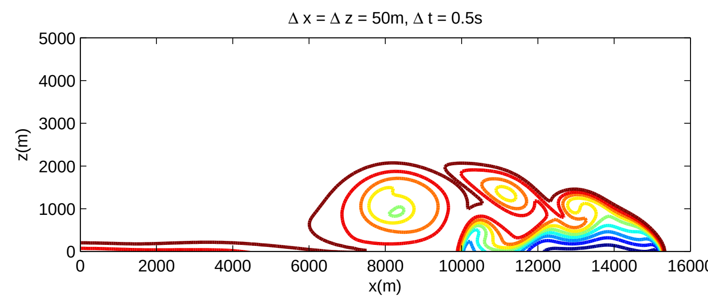
$$\Delta x = 200m$$



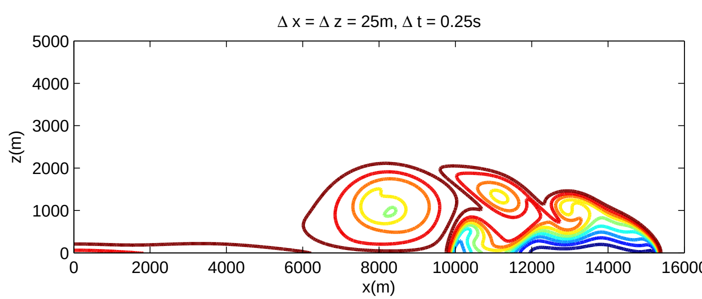
$$\Delta x = 100m,$$



$$\Delta x = 50m$$

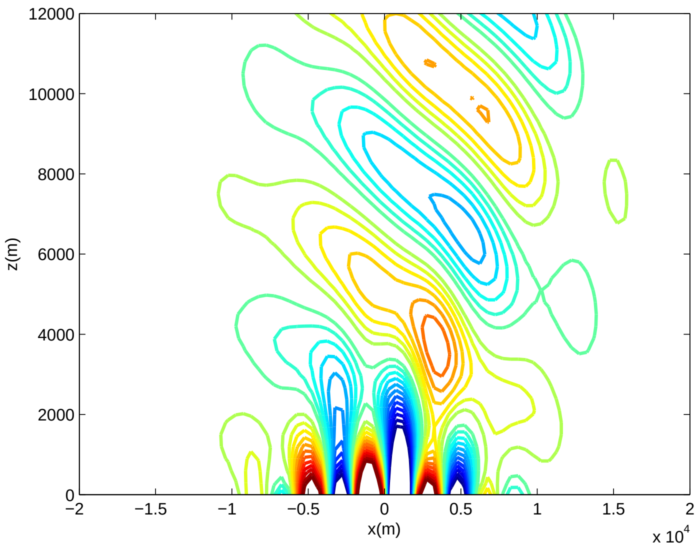


$$\Delta x = 25m$$



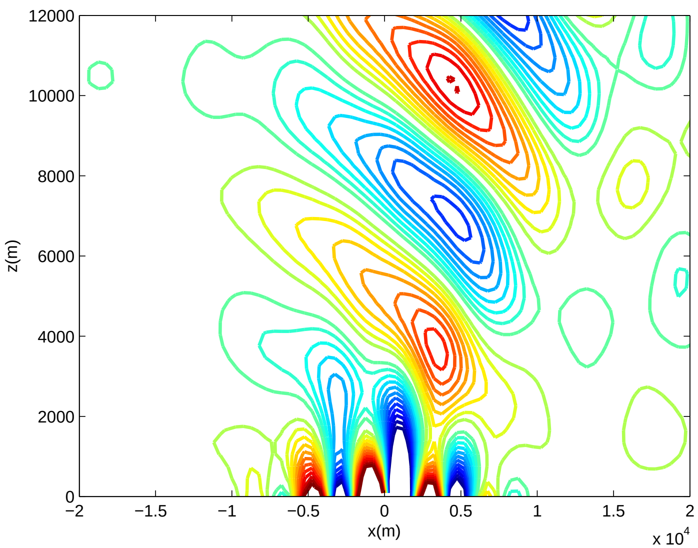
Schär hill test

- Linear Analytic solution:

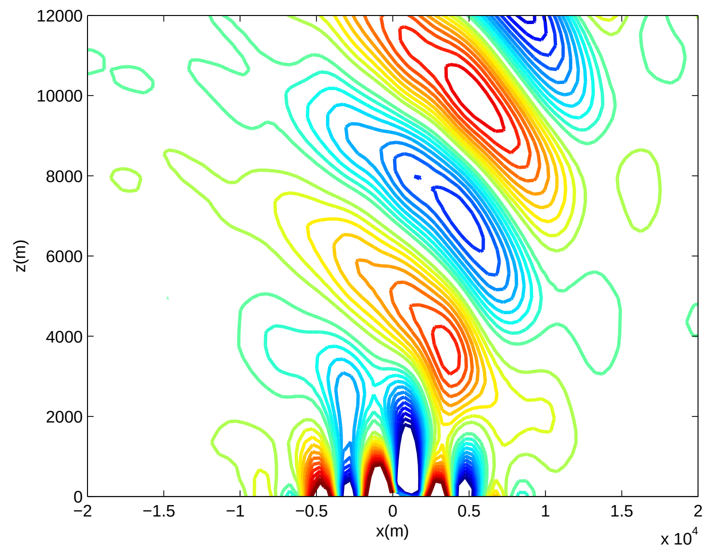


Schär hill test

- Mass-Conserving mode:



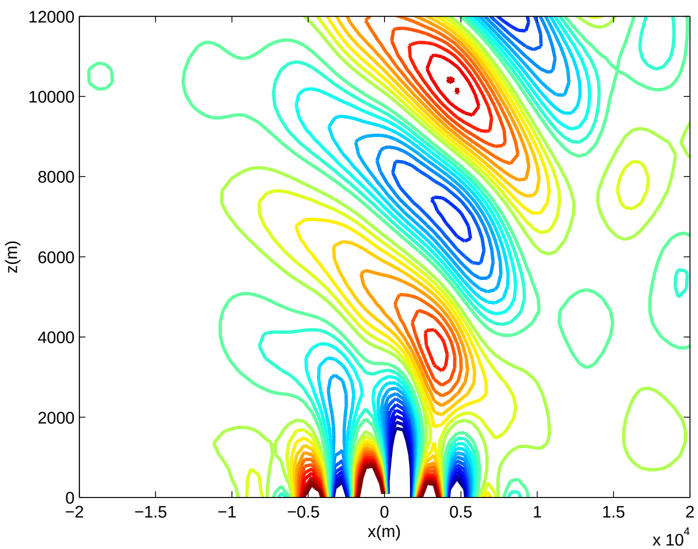
$\Delta t = 8s,$



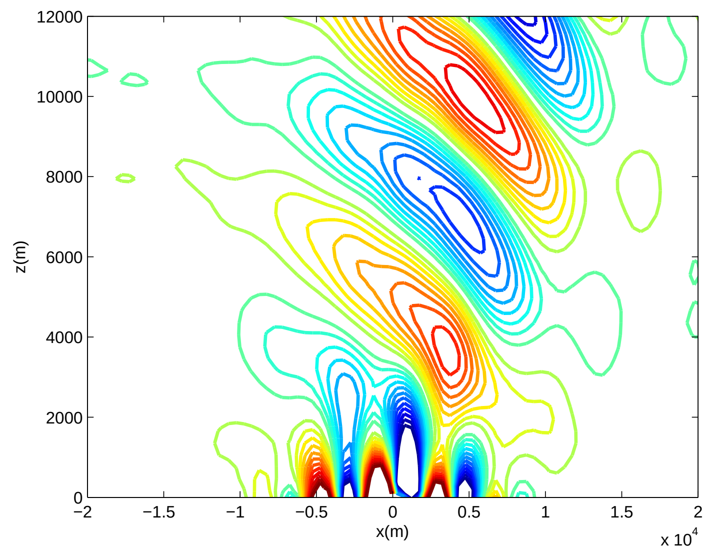
$\Delta t = 40s$

Schär hill test

- Standard SISL mode:



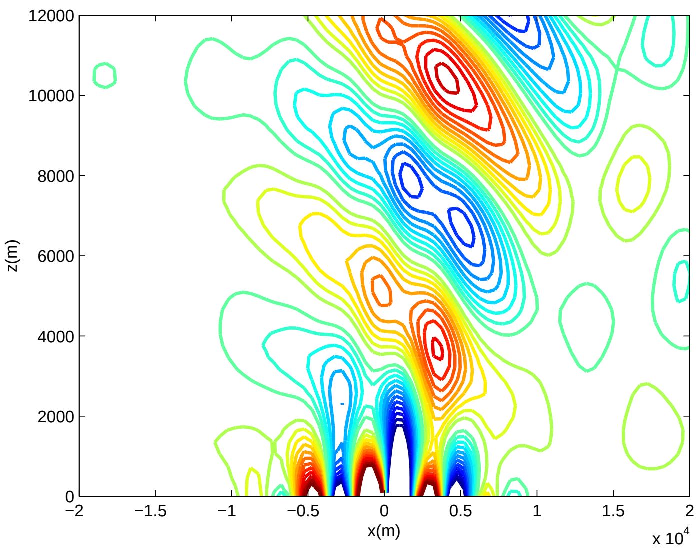
$\Delta t = 8s,$



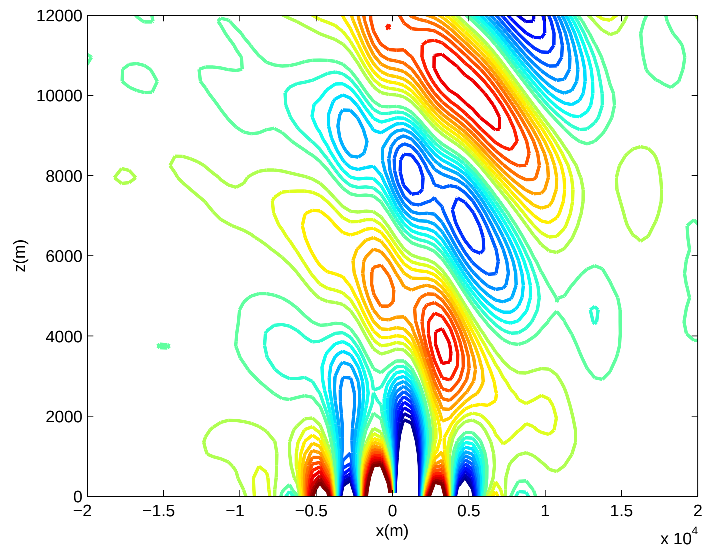
$\Delta t = 40s$

Schär hill test

- Mass-Conserving mode with Eulerian $\dot{\eta}$ formulation:



$\Delta t = 8s,$



$\Delta t = 40s$

Conclusions

- Mass conserving algorithm (SLICE) has been coupled to a SL scheme
- Results for mass conserving mode are as good as standard SISL scheme and compare well to published results
- Lagrangian formulation of $\dot{\eta}$ is important for numerical consistency.
- Remains to test coupling of the full 3D nonhydrostatic equations.

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References

1. Melvin *et al.* *"An inherently mass-conserving semi-implicit semi-Lagrangian discretisation of the nonhydrostatic vertical slice equations"* - Submitted to QJRMS.

2. Zerroukat *et al.* *"An inherently mass-conserving semi-implicit semi-Lagrangian discretisation of the shallow water equations on the sphere"* - QJRMS **135**: 1104-1116.

3. Zerroukat *et al.* *"SLICE: A Semi-Lagrangian Inherently Conserving and Efficient scheme for transport problems"* - QJRMS **128**: 2801-2820.

4. Standard test set for 2D vertical slice dynamical cores: *http://box.mmm.ucar.edu/projects/srnwp_tests/index.html*.