A proposed modification to the Robert-Asselin time filter

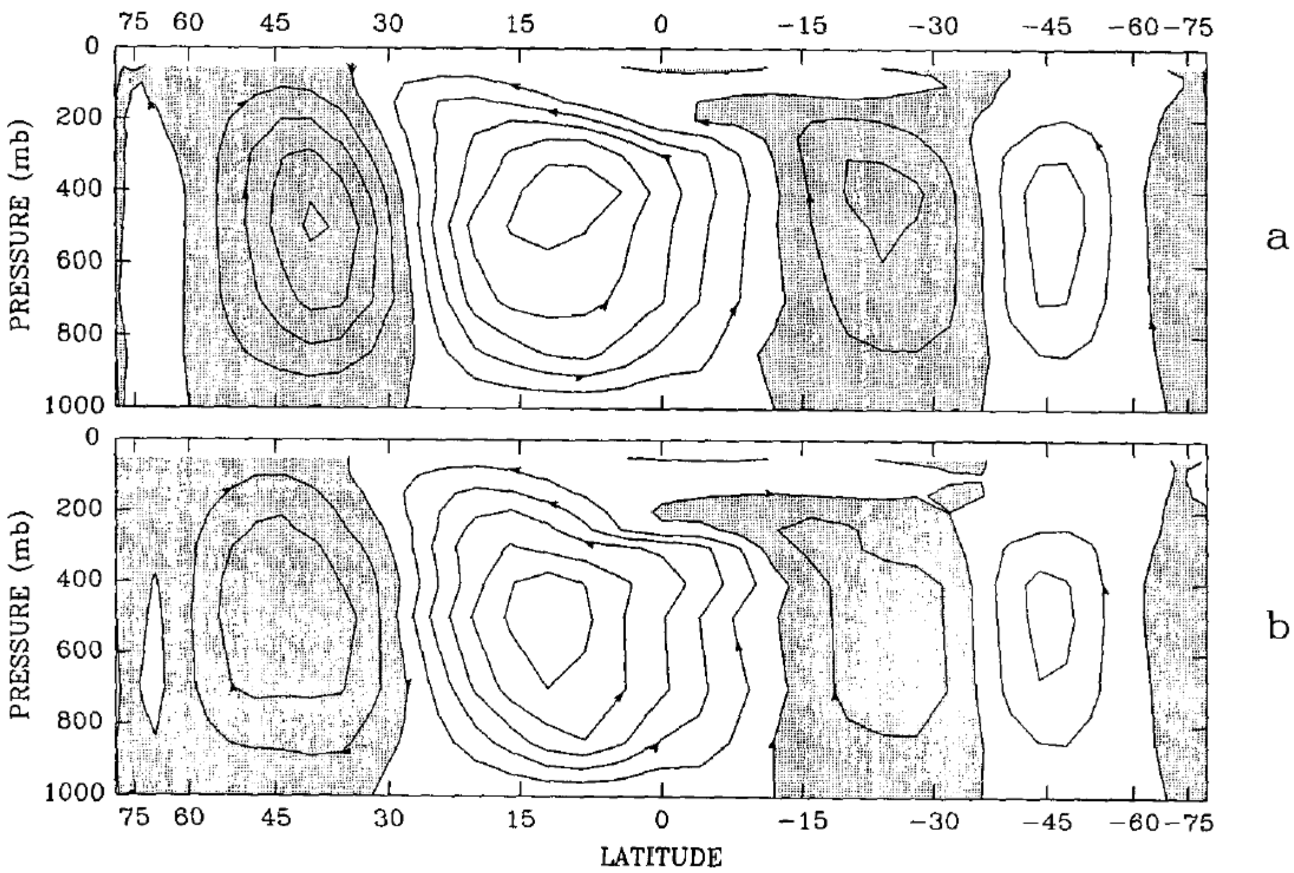
Paul Williams

Department of Meteorology, University of Reading, UK

Sources of error in weather and climate predictions

- uncertain initial state
- uncertain boundary conditions and forcing
- model error
  - physical and dynamical assumptions
  - parameterisation of sub-gridscale processes
  - discrete time stepping

Impact of different time-stepping schemes in an atmosphere GCM



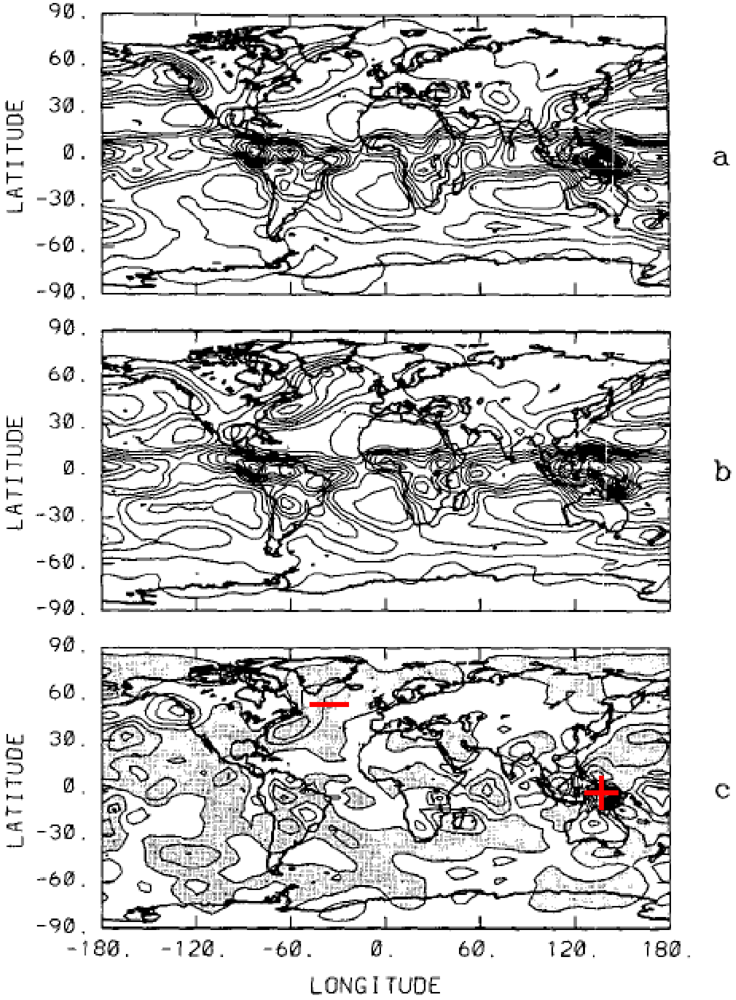
leapfrog

Matsuno

FIG. 6. Eddy-induced streamfunction ψ for the period 1 January–28 February 1979, obtained by solving (39) using the history tapes from the simulation with (a) the leapfrog scheme and (b) the Matsuno scheme. Contour interval 20 m mb s^{-1}.

(Pfeffer et al. 1992)

Impact of different time-stepping schemes in an atmosphere GCM



leapfrog

Matsuno

leapfrog – Matsuno

FIG. 10. Mean precipitation rates for the period 1 January–28 February 1979, simulated with (a) the leapfrog scheme and (b) the Matsuno scheme; (c) the difference between the two (leapfrog minus Matsuno). Contour interval is 2 mm day^{-1}. Shading represents negative values. The data have been smoothed over nine grid points on a 4° latitude × 5° longitude grid.

(Pfeffer et al. 1992)

Impact of different time-stepping schemes in an atmosphere GCM

"These results give evidence that climate simulations are sensitive not only to physical parameterizations of subgrid-scale processes but also to the numerical methodology employed."

(Pfeffer et al. 1992)

Leapfrog time stepping with a Robert-Asselin filter (LF+RA)

- Widely used in current numerical models
  - atmosphere: ECHAM, CAM, MESO-NH, HIRLAM, COSMO, RAMS, FSU-GSM, FSU-NRSM, NCEP-GFS, NCEP-RSM, KMCM, LIMA, SPEEDY, IGCM, PUMA
  - ocean: NEMO, GFDL-MOM, POM, MICOM, HYCOM, ICON, OFES
  - others: QUAGMIRE, MORALS, SAM, ARPS, CASL, CReSS, JTGCM, ECOMSED, UKMO-LEM, MPI-REMO, GTM

- Asselin (1972) has received over 450 citations
  - 300 in atmospheric science journals
  - 100 in oceanography journals
  - 50 in fluid mechanics journals

- "The Robert-Asselin filter has proved immensely popular, and has been widely used for over 20 years. However, it is not the last word…" (Lynch 1991)

LF+RA
(1st order)

proposed
modification
to LF+RA
(3rd order)

TABLE 1. Comparison of time differencing schemes. The amplitude, phase speed, and time step limitations are those associated with the application of each scheme to the oscillation Eq. (4). Storage and efficiency factors are defined in the text. Here, $h = \Delta t$ and $p = \omega h$.

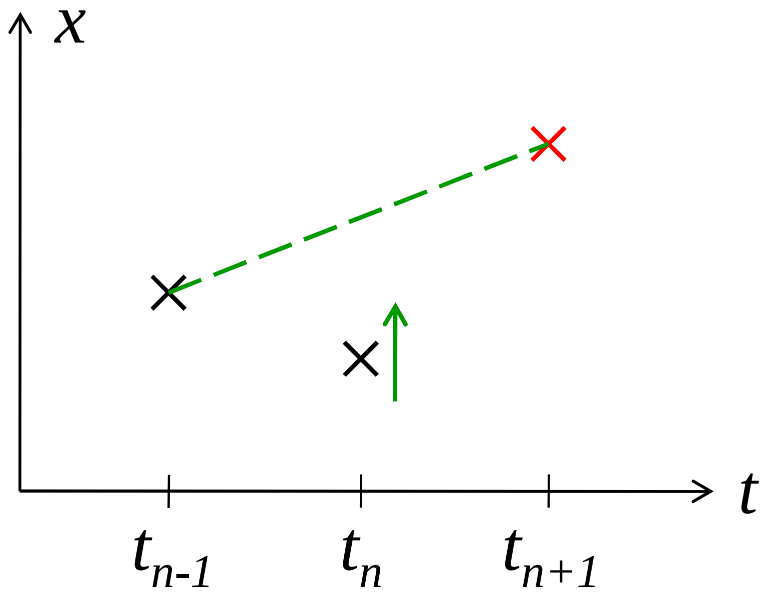
Method	Order	Formula	Storage factor	Efficiency factor	Amplitude error	Phase error	Maximum $\omega\Delta t$
Forward (Adams–Bashforth)	1	$\phi^{n+1} = \phi^n + hF(\phi^n)$	2	0	$1 + \frac{p^2}{2}$	$1 - \frac{p^2}{3}$	0
Backward (Adams–Moulton)	1	$\phi^{n+1} = \phi^n + hF(\phi^{n+1})$	Implicit	∞	$1 - \frac{p^2}{2}$	$1 - \frac{p^2}{3}$	∞
Matsuno	1	$\phi^* = \phi^n + hF(\phi^n)$ $\phi^{n+1} = \phi^n + hF(\phi^*)$	2	.5	$1 - \frac{p^2}{2}$	$1 + \frac{2}{3}p^2$	1
Asselin-filtered leapfrog	1	$\phi^{n+1} = \overline{\phi^{n-1}} + 2hF(\phi^n)$ $\overline{\phi^n} = \phi^n + \gamma(\overline{\phi^{n-1}} - 2\phi^n + \phi^{n+1})$	3	<1	$1 - \frac{\gamma}{2(1-\gamma)}p^2$	$1 + \frac{1+2\gamma}{6(1-\gamma)}p^2$	<1
Leapfrog	2	$\phi^{n+1} = \phi^{n-1} + 2hF(\phi^n)$	2	1	1	$1 + \frac{p^2}{6}$	1
Runge–Kutta (Williamson/Huen)	2	$q_1 = hF(\phi^n), \quad \phi_1 = \phi^n + q_1$ $q_2 = hF(\phi_1) - q_1, \quad \phi^{n+1} = \phi_1 + q_2/2$	2	0	$1 + \frac{p^4}{8}$	$1 + \frac{p^2}{6}$	0
Adams–Bashforth	2	$\phi^{n+1} = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$	3	0	$1 + \frac{p^4}{4}$	$1 + \frac{5}{12}p^2$	0
Adams–Moulton (Trapizoidal)	2	$\phi^{n+1} = \phi^n + \frac{h}{2}[F(\phi^{n+1}) + F(\phi^n)]$	Implicit	∞	1	$1 - \frac{p^2}{12}$	∞
Leapfrog, then Adams–Bashforth (Magazenkov)	2	$\phi^n = \phi^{n-2} + 2hF(\phi^{n-1})$ $\phi^{n+1} = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$	3	.67	$1 - \frac{p^4}{4}$	$1 + \frac{p^2}{6}$	.67
Leapfrog-Trapizoidal (Kurihara)	2	$\phi^* = \phi^{n-1} + 2hF(\phi^n)$ $\phi^{n+1} = \phi^n + \frac{h}{2}[F(\phi^n) + F(\phi^*)]$	3	.71	$1 - \frac{p^4}{4}$	$1 - \frac{p^2}{12}$	1.41
Young's method A	2	$\phi_1 = \phi^n + hF(\phi^n)/2$ $\phi_2 = \phi_1 + hF(\phi_1)/2$ $\phi^{n+1} = \phi^n + hF(\phi_2)$	3	0	$1 + \frac{p^6}{128}$	$1 + \frac{p^2}{24}$	0
Runge–Kutta (Williamson)	3	$q_1 = hF(\phi^n), \quad \phi_1 = \phi^n + q_1/3,$ $q_2 = hF(\phi_1) - 5q_1/9, \quad \phi_2 = \phi^1 + 15q_2/16$ $q_3 = hF(\phi_2) - 153q_2/128, \quad \phi^{n+1} = \phi_2 + 8q_3/15$	2	.58	$1 - \frac{p^4}{24}$	$1 + \frac{p^4}{30}$	1.73
ABM predictor-corrector	3	$\phi^* = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$ $\phi^{n+1} = \phi^* + \frac{5h}{12}[F(\phi^*) - 2F(\phi^n) + F(\phi^{n-1})]$	4	.60	$1 - \frac{19}{144}p^4$	$1 + \frac{1243}{8640}p^4$	1.20
Adams–Moulton	3	$\phi^{n+1} = \phi^n + \frac{h}{12}[5F(\phi^{n+1}) + 8F(\phi^n) - F(\phi^{n-1})]$	Implicit	0	$1 + \frac{p^4}{24}$	$1 - \frac{11}{720}p^4$	0
Adams–Bashforth	3	$\phi^{n+1} = \phi^n + \frac{h}{12}[23F(\phi^n) - 16F(\phi^{n+1}) + 5F(\phi^{n-2})]$	4	.72	$1 - \frac{3}{8}p^4$	$1 + \frac{289}{720}p^4$	0.72
Runge–Kutta (Classical)	4	$k_0 = hF(\phi^n)$ $k_1 = hF(\phi^n + k_0/2)$ $k_2 = hF(\phi^n + k_1/2)$ $k_3 = hF(\phi^n + k_2)$ $\phi^{n+1} = \phi^n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$	3*	.70	$1 - \frac{p^6}{144}$	$1 - \frac{p^4}{120}$	2.82
ABM predictor-corrector	4	$\phi^* = \phi^n + \frac{h}{12}[23F(\phi^n) - 16F(\phi^{n-1}) + 5F(\phi^{n-2})]$ $\phi^{n+1} = \phi^* + \frac{3h}{8}[F(\phi^*) - 3F(\phi^n) + 3F(\phi^{n-1}) - F(\phi^{n-2})]$	5	.59	$1 - \frac{265}{1536}p^6$	$1 - \frac{329}{2880}p^4$	1.18
Adams–Moulton	4	$\phi^{n+1} = \phi^n + \frac{h}{24}[9F(\phi^{n+1}) + 19F(\phi^n) - 5F(\phi^{n-1}) + F(\phi^{n-2})]$	Implicit	0	$1 + \frac{p^6}{48}$	$1 + \frac{19}{720}p^4$	0
Adams–Bashforth	4	$\phi^{n+1} = \phi^n + \frac{h}{24}[55F(\phi^n) - 59F(\phi^{n-1}) + 37F(\phi^{n-2}) - 9F(\phi^{n-3})]$	5	.43	$1 - \frac{13}{24}p^6$	$1 - \frac{251}{720}p^4$	.43

increasing accuracy

(Durran 1991)

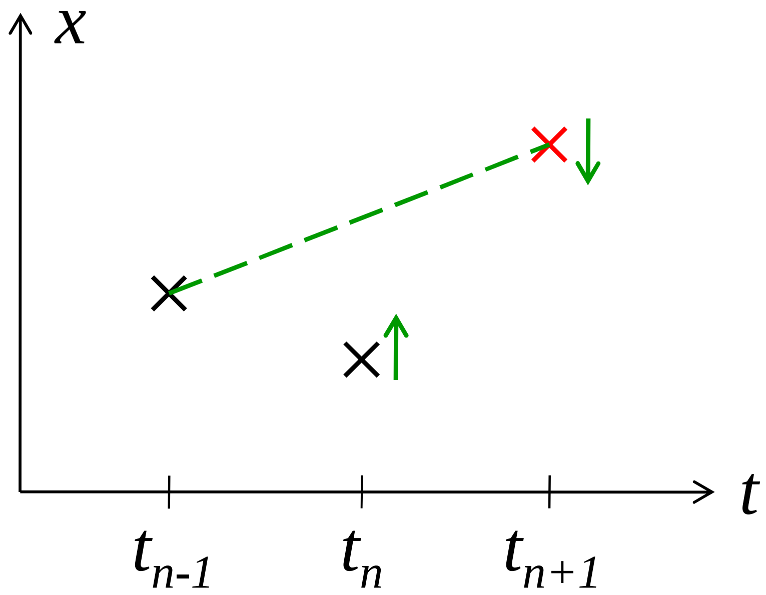
The proposed modified scheme

<u>standard scheme due to Robert (1966) and Asselin (1972)</u>



- use leapfrog to calculate x_{n+1}
- nudge x_n
- reduces curvature <u>but does not conserve mean</u>
- only first-order accurate

<u>modification proposed by Williams (2009)</u>



- use leapfrog to calculate x_{n+1}
- nudge x_n <u>and</u> x_{n+1}
- reduces curvature <u>and conserves mean</u>
- third-order accurate

The proposed modified scheme

Simple test integration

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\omega Y$$

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = +\omega X$$

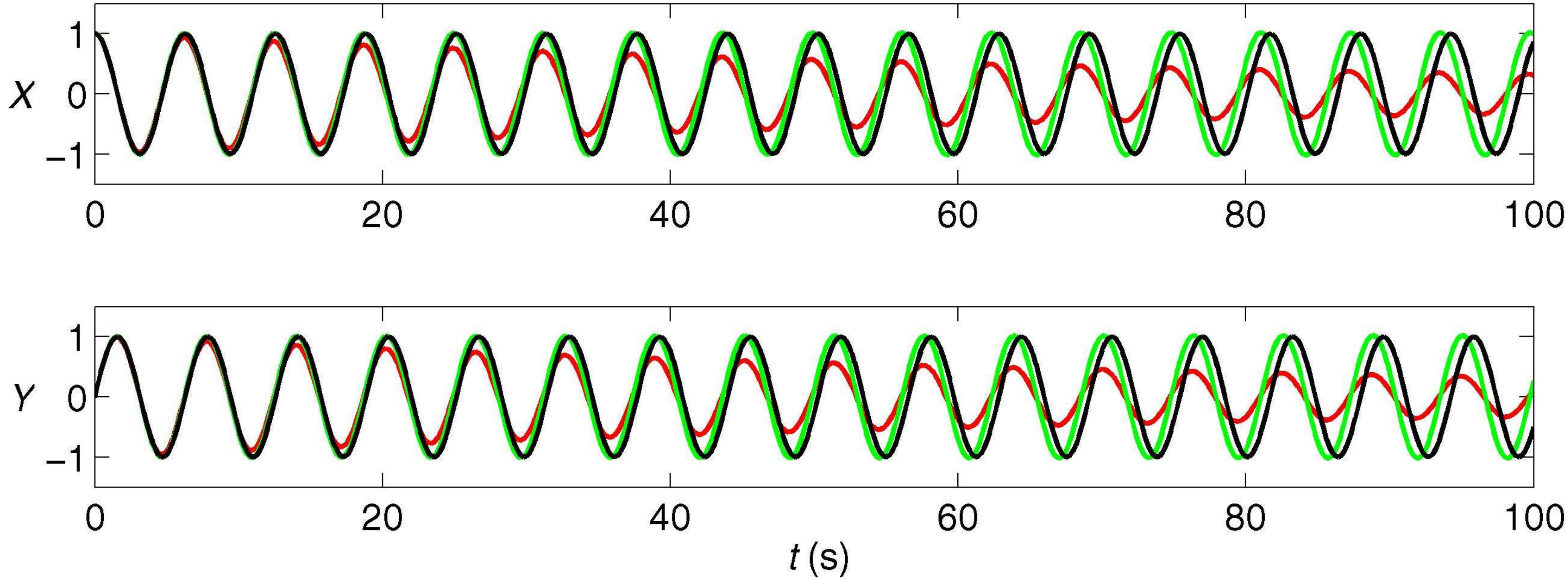
$$\omega = 1 \text{ rad s}^{-1}$$

exact

standard

modified

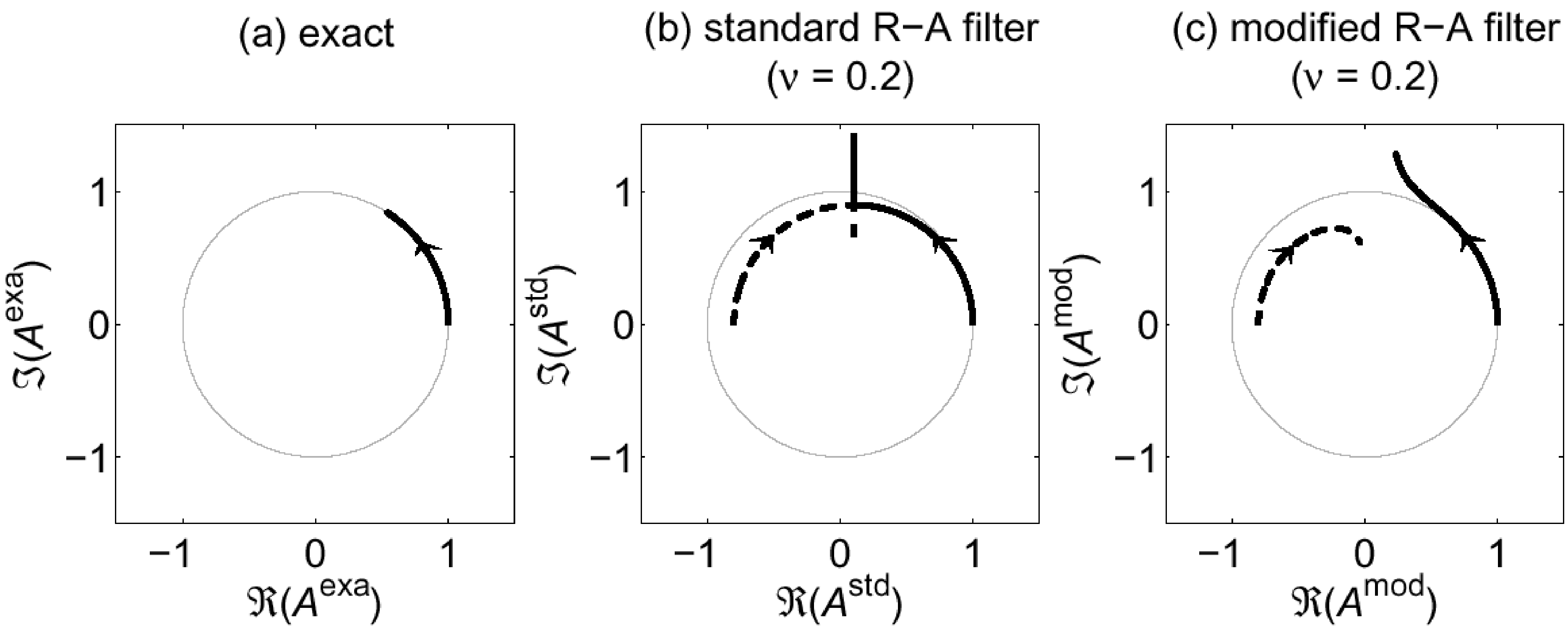
$$\left.\begin{array}{l}\textbf{standard}\\ \textbf{modified}\end{array}\right\} \begin{array}{c}\Delta t = 0.2 \text{ s}\\ \nu = 0.2\end{array}$$



(Williams 2009)

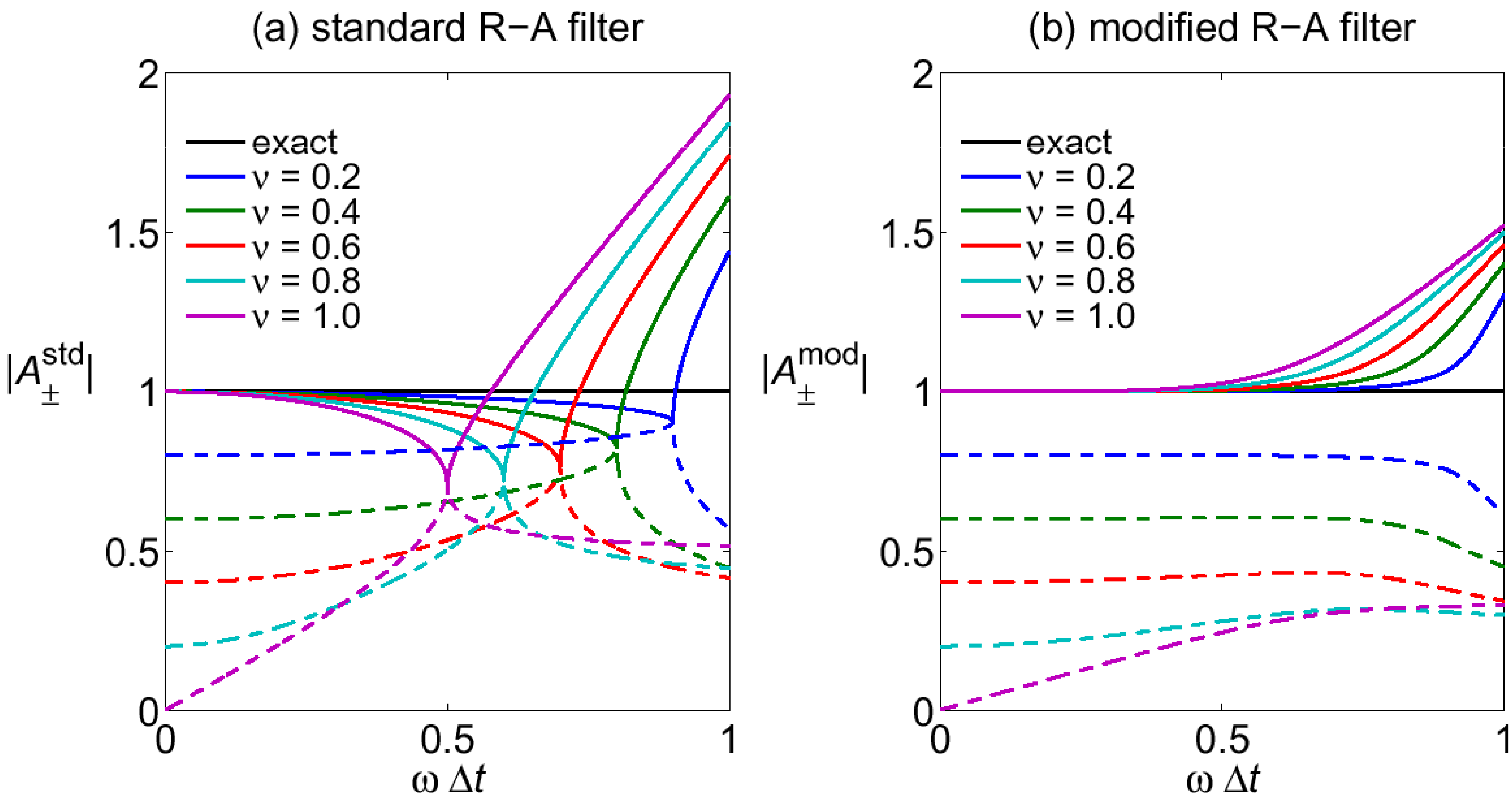
Analysis: amplification factor

Let $\dot{F} = i\omega F$ and $A = F(t+\Delta t)\ /\ F(t)$ and trace A as $\omega\Delta t = 0 \rightarrow 1$:



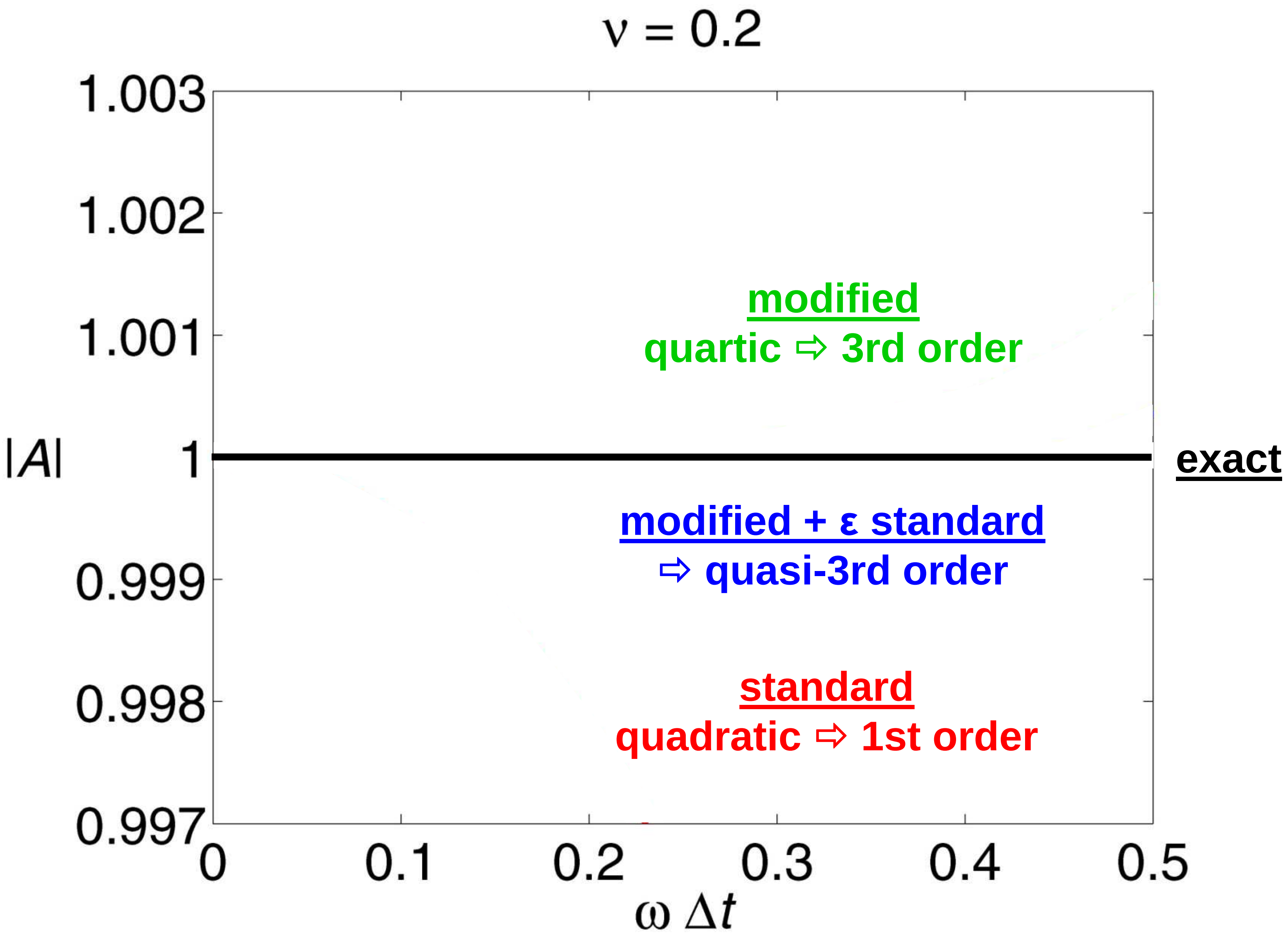
(Williams 2009)

Analysis: amplification factor



(Williams 2009)

Analysis: amplification factor



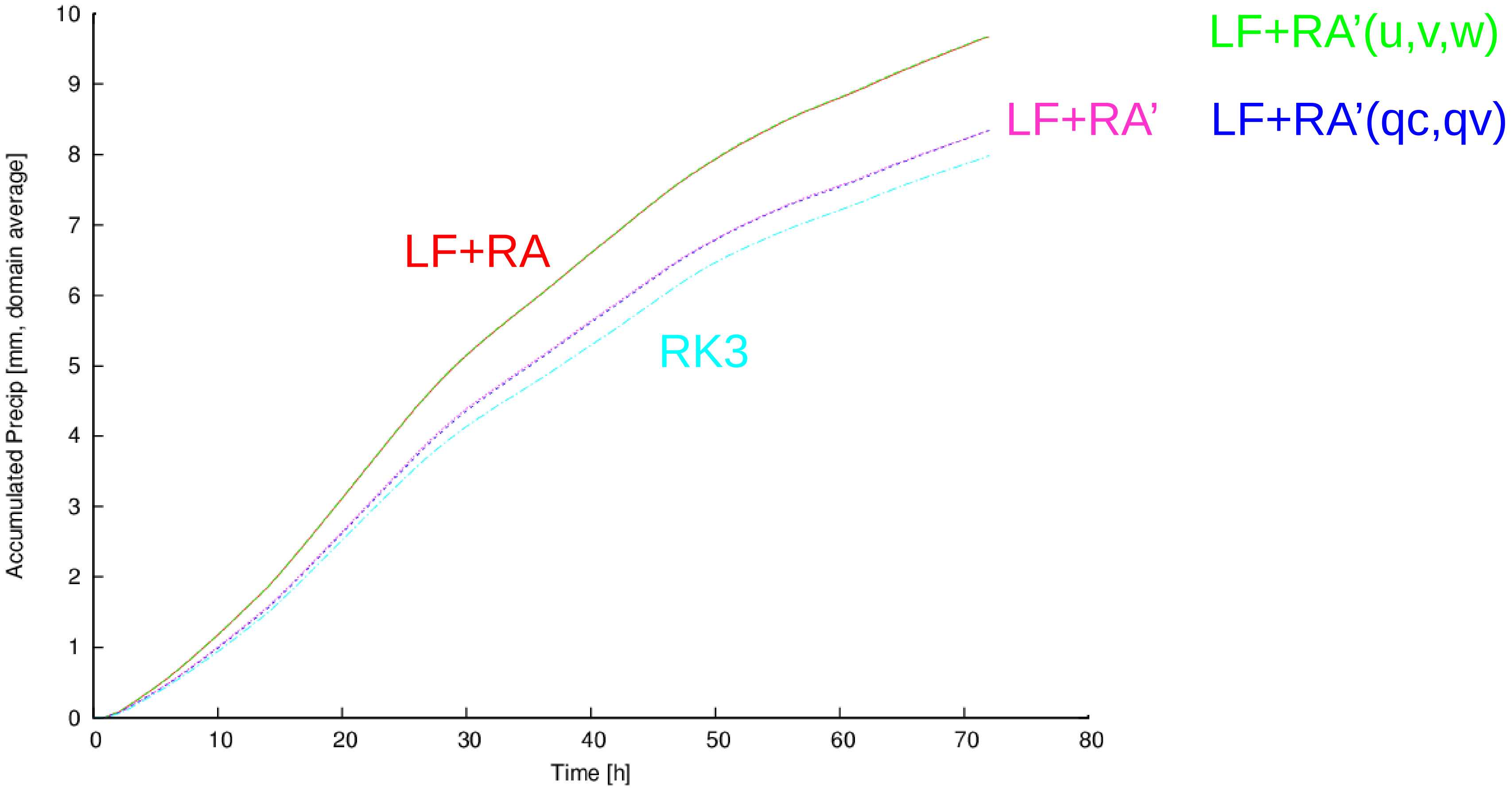
(Williams 2009)

Implementation in existing code

```
! Compute tendency at this time step
tendency = [⋯]

! Leapfrog step
F_next = F_last + tendency*2*delta_t

! Compute filter displacement
d = nu/2*(F_last - 2*F_this + F_next)

! Apply filter
F this = F this + d*alp
```

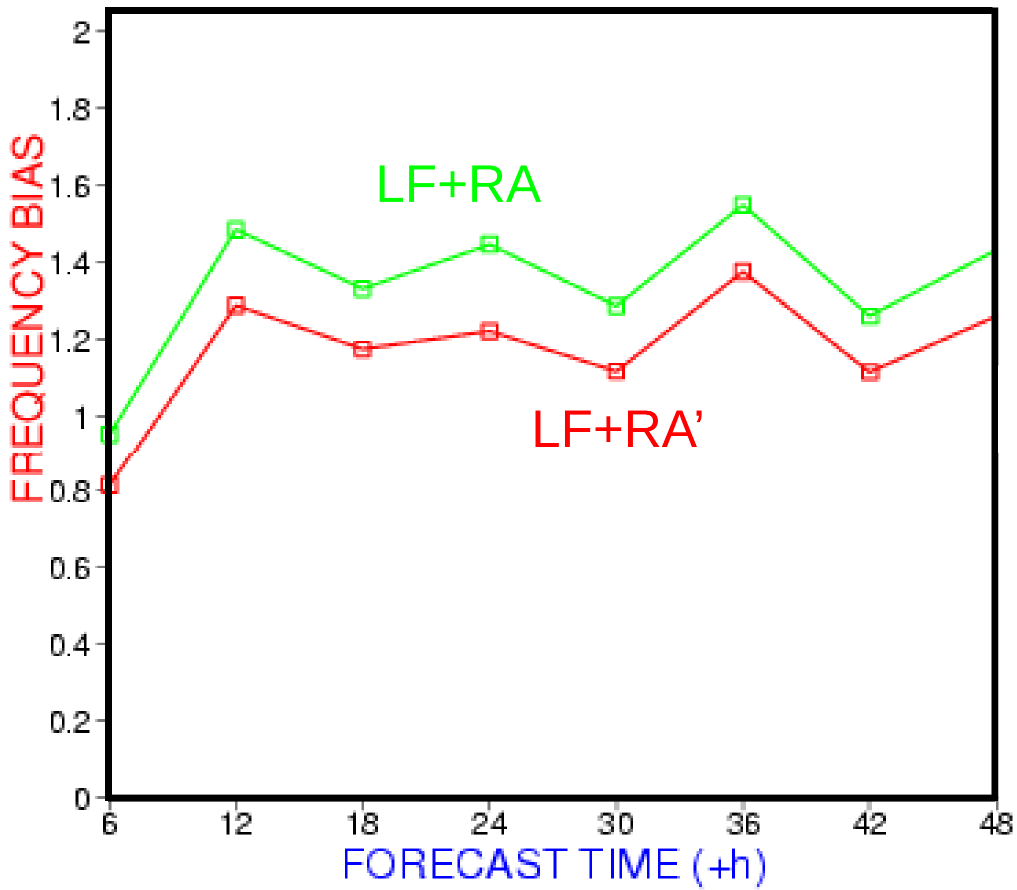
Implementation in COSMO-7

- Thanks to Oliver Fuhrer at MeteoSwiss
- Test case: 72h forecast over Europe starting at 00Z on 9 Feb 2009
- The modified filter significantly reduces precipitation…
- …and all that matters is whether or not it is applied to the moisture variables!



Implementation in COSMO-ME

- Thanks to Lucio Torrisi at CNMCA, Italy
- Statistical analysis: two 48h forecasts per day from 16 Dec 2008 to 18 Jan 2009
- The modified filter significantly improves precipitation forecasts (2-10 mm / 6h)
- There are no significant changes in other quantities



Summary

- The leapfrog time-stepping scheme and Robert-Asselin filter are widely used in atmosphere and ocean numerical models
- A simple proposed modification to the filter greatly increases the accuracy…
- …at virtually no extra computational cost…
- …and is very easy to implement!
- All the above statements remain true for semi-implicit integrations

Reference

Williams, PD (2009)

A proposed modification to the Robert-Asselin time filter

Monthly Weather Review **137**(8), pp 2538-2546.

p.d.williams@reading.ac.uk

www.met.reading.ac.uk/~williams