A proposed modification to the Robert-Asselin time filter

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Sources of error in weather and climate predictions

• uncertain initial state
• uncertain boundary conditions and forcing
• model error
  – physical and dynamical assumptions
  – parameterisation of sub-gridscale processes
  – discrete time stepping
Impact of different time-stepping schemes in an atmosphere GCM

FIG. 6. Eddy-induced streamfunction $\psi$ for the period 1 January–28 February 1979, obtained by solving (39) using the history tapes from the simulation with (a) the leapfrog scheme and (b) the Matsuno scheme. Contour interval 20 m mb s$^{-1}$.

(Pfeffer et al. 1992)
Impact of different time-stepping schemes in an atmosphere GCM

leapfrog

Matsuno

leapfrog – Matsuno

Fig. 10. Mean precipitation rates for the period 1 January–28 February 1979, simulated with (a) the leapfrog scheme and (b) the Matsuno scheme; (c) the difference between the two (leapfrog minus Matsuno). Contour interval is 2 mm day$^{-1}$. Shading represents negative values. The data have been smoothed over nine grid points on a 4° latitude × 5° longitude grid.

(Pfeffer et al. 1992)
Impact of different time-stepping schemes in an atmosphere GCM

“These results give evidence that climate simulations are sensitive not only to physical parameterizations of subgrid-scale processes but also to the numerical methodology employed.”

(Pfeffer et al. 1992)
Leapfrog time stepping with a Robert-Asselin filter (LF+RA)

- Widely used in current numerical models
  - **atmosphere**: ECHAM, CAM, MESO-NH, HIRLAM, COSMO, RAMS, FSU-GSM, FSU-NRSM, NCEP-GFS, NCEP-RSM, KMCM, LIMA, SPEEDY, IGCM, PUMA
  - **ocean**: NEMO, GFDL-MOM, POM, MICOM, HYCOM, ICON, OFES
  - **others**: QUAGMIRE, MORALS, SAM, ARPS, CASL, CReSS, JTGCM, ECOMSED, UKMO-LEM, MPI-REMO, GTM

- Asselin (1972) has received over 450 citations
  - 300 in **atmospheric science** journals
  - 100 in **oceanography** journals
  - 50 in **fluid mechanics** journals

- “The Robert-Asselin filter has proved immensely popular, and has been widely used for over 20 years. However, it is not the last word…” (Lynch 1991)
<table>
<thead>
<tr>
<th>Method</th>
<th>Order</th>
<th>Formula</th>
<th>Storage factor</th>
<th>Efficiency factor</th>
<th>Amplitude error</th>
<th>Phase error</th>
<th>Maximum α/dt</th>
</tr>
</thead>
<tbody>
<tr>
<td>Forward (Adams-Bashforth)</td>
<td>1</td>
<td>$\phi^{n+1} = \phi^n + hF(\phi^n)$</td>
<td>2</td>
<td>0</td>
<td>$1 - \frac{h^2}{2}$</td>
<td>1</td>
<td>0</td>
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<tr>
<td>Backward (Adams-Moulton)</td>
<td>1</td>
<td>$\phi^{n+1} = \phi^n + hF(\phi^{n+1})$</td>
<td>Implicit</td>
<td>$\infty$</td>
<td>$1 - \frac{h^2}{2}$</td>
<td>$1 - \frac{h^2}{3}$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>Matsuno</td>
<td>2</td>
<td>$\phi^* = \phi^n + hF(\phi^n)$</td>
<td>2</td>
<td>$.5$</td>
<td>$1 - \frac{h^2}{2}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Adam's-forman leapfrog</td>
<td>1</td>
<td>$\phi^{n+1} = \phi^n - \frac{h^2}{2} (\frac{3}{2} F(\phi^n) + \frac{1}{2} \frac{d^2 \phi}{dt^2})$</td>
<td>3</td>
<td>$.7$</td>
<td>$1 - \frac{7}{24} h^2$</td>
<td>$1 + \frac{49 h^4}{240}$</td>
<td>$.7$</td>
</tr>
<tr>
<td>Leapfrog</td>
<td>2</td>
<td>$\phi^{n+1} = \phi^n + \frac{h^2}{2} F(\phi^n)$</td>
<td>2</td>
<td>1</td>
<td>$1 + \frac{4 h^2}{5}$</td>
<td>1</td>
<td>1</td>
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<tr>
<td>Runge-Kutta (Williamson)</td>
<td>2</td>
<td>$\phi_{n+1} = \frac{2}{3} \phi_n + \frac{1}{3} \phi_{n-1}$</td>
<td>2</td>
<td>0</td>
<td>$\frac{h^2}{6}$</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Adams-Bashforth</td>
<td>2</td>
<td>$\phi^{n+1} = \phi^n + \frac{h}{2} F(\phi^n)$</td>
<td>3</td>
<td>0</td>
<td>$\frac{h^2}{12}$</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Adams-Moulton (Trapezoidal)</td>
<td>2</td>
<td>$\phi^{n+1} = \phi^n + \frac{h}{2} (F(\phi^n) + F(\phi^{n+1}))$</td>
<td>Implicit</td>
<td>0</td>
<td>$\frac{h^2}{12}$</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Leapfrog, then Adams-Bashforth (Magenichev)</td>
<td>2</td>
<td>$\phi^{n+1} = \phi^n + \frac{1}{6} (F(\phi^n) + F(\phi^{n+1}))$</td>
<td>3</td>
<td>$.71$</td>
<td>$1 - \frac{h^2}{4}$</td>
<td>$1 - \frac{h^2}{12}$</td>
<td>$1.41$</td>
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<tr>
<td>Leapfrog-Trapezoidal (Kuruhi)</td>
<td>2</td>
<td>$\phi^{n+1} = \phi^n + \frac{1}{6} (F(\phi^n) + F(\phi^{n+1}))$</td>
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<td>0</td>
<td>$\frac{h^2}{12}$</td>
<td>0</td>
<td>0</td>
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<tr>
<td>Young's method A</td>
<td>2</td>
<td>$\phi_{n+1} = \phi_n + \frac{1}{3} (\phi_{n+1} + \phi_n)$</td>
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<td>0</td>
<td>$\frac{h^2}{12}$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Runge-Kutta (Williamson)</td>
<td>3</td>
<td>$\phi_{n+1} = \frac{3}{4} \phi_n + \frac{1}{4} \phi_{n-1}$</td>
<td>2</td>
<td>$.58$</td>
<td>$\frac{h^2}{40}$</td>
<td>$1 + \frac{h^2}{24}$</td>
<td>1.73</td>
</tr>
<tr>
<td>AIM predictor-corrector</td>
<td>3</td>
<td>$\phi^{n+1} = \phi^n + \frac{h^2}{12} [F(\phi^n) - F(\phi^{n+1})]$</td>
<td>4</td>
<td>.68</td>
<td>$1 - \frac{19}{144} h^2$</td>
<td>$1 + \frac{1343}{8640} h^4$</td>
<td>1.80</td>
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<tr>
<td>Adams-Moulton</td>
<td>3</td>
<td>$\phi^{n+1} = \phi^n + \frac{h}{12} (3F(\phi^n) - 10F(\phi^{n+1}) + 3F(\phi^{n+2}))$</td>
<td>Implicit</td>
<td>0</td>
<td>$\frac{h^2}{24}$</td>
<td>$1 - \frac{13}{720} h^2$</td>
<td>9</td>
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<tr>
<td>Adams-Bashforth</td>
<td>3</td>
<td>$\phi^{n+1} = \phi^n + \frac{h}{12} (23F(\phi^n) - 16F(\phi^{n+1}) + 5F(\phi^{n+2}))$</td>
<td>4</td>
<td>.72</td>
<td>$\frac{h^2}{8}$</td>
<td>$1 + \frac{289}{720} h^4$</td>
<td>0.72</td>
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<tr>
<td>Runge-Kutta (Classical)</td>
<td>4</td>
<td>$\phi^{n+1} = \phi^n + \frac{h}{12} (3F(\phi^n) - 4F(\phi^{n+1}) + 2F(\phi^{n+2}) - F(\phi^{n+3})$</td>
<td>3</td>
<td>$.7$</td>
<td>$\frac{h^2}{144}$</td>
<td>$1 - \frac{h^2}{120}$</td>
<td>2.82</td>
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<tr>
<td>AIM predictor-corrector</td>
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<td>$\phi^{n+1} = \phi^n + \frac{h}{12} (23F(\phi^n) - 16F(\phi^{n+1}) + 5F(\phi^{n+2}) + 3F(\phi^{n+3}))$</td>
<td>3</td>
<td>.72</td>
<td>$\frac{h^2}{120}$</td>
<td>$1 - \frac{h^2}{240}$</td>
<td>1.44</td>
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<tr>
<td>Adams-Moulton</td>
<td>4</td>
<td>$\phi^{n+1} = \phi^n + \frac{h}{24} (6F(\phi^n) - 20F(\phi^{n+1}) + 10F(\phi^{n+2}) - 5F(\phi^{n+3}) + F(\phi^{n+4}))$</td>
<td>Implicit</td>
<td>0</td>
<td>$\frac{h^2}{48}$</td>
<td>$1 + \frac{19}{720} h^4$</td>
<td>0</td>
</tr>
<tr>
<td>Adams-Bashforth</td>
<td>4</td>
<td>$\phi^{n+1} = \phi^n + \frac{h}{24} (23F(\phi^n) - 16F(\phi^{n+1}) + 3F(\phi^{n+2}) - F(\phi^{n+3}) + F(\phi^{n+4}))$</td>
<td>4</td>
<td>.43</td>
<td>$\frac{h^2}{24}$</td>
<td>$1 - \frac{13}{720} h^2$</td>
<td>.43</td>
</tr>
</tbody>
</table>
The proposed modified scheme

standard scheme due to Robert (1966) and Asselin (1972)

- use leapfrog to calculate $x_{n+1}$
- nudge $x_n$
- reduces curvature but does not conserve mean
- only first-order accurate

modification proposed by Williams (2009)

- use leapfrog to calculate $x_{n+1}$
- nudge $x_n$ and $x_{n+1}$
- reduces curvature and conserves mean
- third-order accurate
The proposed modified scheme

\[ x \]

leapfrog  filter
Simple test integration

\[
\begin{align*}
\frac{dX}{dt} &= -\omega Y \\
\frac{dY}{dt} &= +\omega X
\end{align*}
\]

\[\omega = 1 \text{ rad s}^{-1}\]

\[
\left\{ \begin{array}{l}
\Delta t = 0.2 \text{ s} \\
\nu = 0.2
\end{array} \right.\]

(Williams 2009)
Analysis: amplification factor

Let $\dot{F} = i\omega F$ and $A = F(t+\Delta t) / F(t)$ and trace $A$ as $\omega \Delta t = 0 \rightarrow 1$:

(a) exact

(b) standard R–A filter ($\nu = 0.2$)

(c) modified R–A filter ($\nu = 0.2$)

(Williams 2009)
Analysis: amplification factor

(a) standard R–A filter

(b) modified R–A filter

\[ |A_{\pm}^{\text{std}}| \]

\[ |A_{\pm}^{\text{mod}}| \]

\( \omega \Delta t \)

(Williams 2009)
Analysis: amplification factor

\( \nu = 0.2 \)

- **modified quartic** \( \Rightarrow \) 3rd order
- **modified + \( \epsilon \) standard** \( \Rightarrow \) quasi-3rd order
- **standard quadratic** \( \Rightarrow \) 1st order

(Williams 2009)
Implementation in existing code

! Compute tendency at this time step
tendency = […]

! Leapfrog step
F_next = F_last + tendency*2*delta_t

! Compute filter displacement
d = nu/2*(F_last - 2*F_this + F_next)

! Apply filter
F_this = F_this + d*alpha
F_next = F_next + d*(alpha-1)
Implementation in COSMO-7

- Thanks to Oliver Fuhrer at MeteoSwiss
- Test case: 72h forecast over Europe starting at 00Z on 9 Feb 2009
- The modified filter significantly reduces precipitation…
- …and all that matters is whether or not it is applied to the moisture variables!
Implementation in COSMO-ME

- Thanks to Lucio Torrisi at CNMCA, Italy
- Statistical analysis: two 48h forecasts per day from 16 Dec 2008 to 18 Jan 2009
- The modified filter significantly improves precipitation forecasts (2-10 mm / 6h)
- There are no significant changes in other quantities
Summary

• The leapfrog time-stepping scheme and Robert-Asselin filter are widely used in atmosphere and ocean numerical models.

• A simple proposed modification to the filter greatly increases the accuracy...

• …at virtually no extra computational cost...

• …and is very easy to implement!

• All the above statements remain true for semi-implicit integrations.
Reference

Williams, PD (2009)
A proposed modification to the Robert-Asselin time filter

*Monthly Weather Review* 137(8), pp 2538-2546.

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