An Improved Third Order Vertical Advection Scheme for the Runge-Kutta Dynamical Core

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Motivation:

in a convection-permitting model (like COSMO-DE) the vertical advection plays a much bigger role than in a convection-parameterising model

→ try to achieve higher accuracy in the vertical advection of dynamic variables (u,v,w,T',p'), too

COSMO-model up to now: vertically implicit centered diff. 2nd order

WRF: vertically explicit upwind scheme (3rd order)

- *advantages:*
  - Fits best to the explicit horizontal advection and the Runge-Kutta-scheme
  - Relatively easy to implement
- disadvantages:
  - Limitation of Courant number: $C_x + C_y + C_z < 1.4$ (*Baldauf, 2008, JCP*) →
    - WRF uses smaller time steps (~15 sec for dx=3km)
    - WRF uses a vertical ‘velocity brake’

→ **Keep the vertically implicit scheme, but try a higher order of approximation**

(COSMO priority project 'Runge-Kutta', Task 8)

Several implicit advection schemes (2-timelevels)

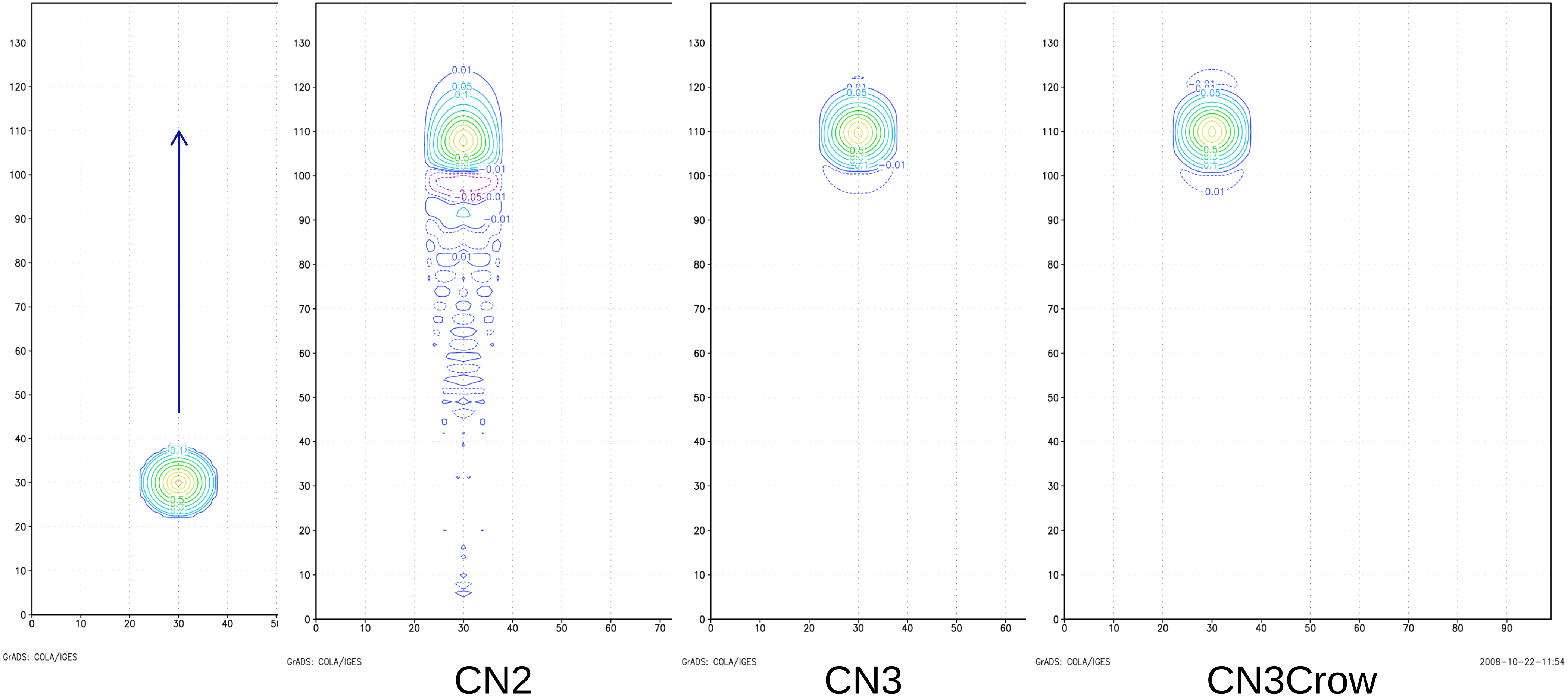
$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \beta A_z(\phi^{n+1}) + (1-\beta) A_z(\phi^n)$$

To achieve higher order in time, too, we will use β=1/2 (‚pure' Crank-Nicholson)

A_z denotes a spatial discretisation of the vertical advection operator:

- Centered differences 2nd order (spatially and temporally) ('*CN2*') current
- Upwind 3rd order (spatially) ('*CN3*') new
- Upwind 3rd order (spatially and temporally) ('*CN3Crow*') new
  its derivation is quite analogous to the explicit Crowley-schemes
  (*Tremback et al., 1987, MWR*)

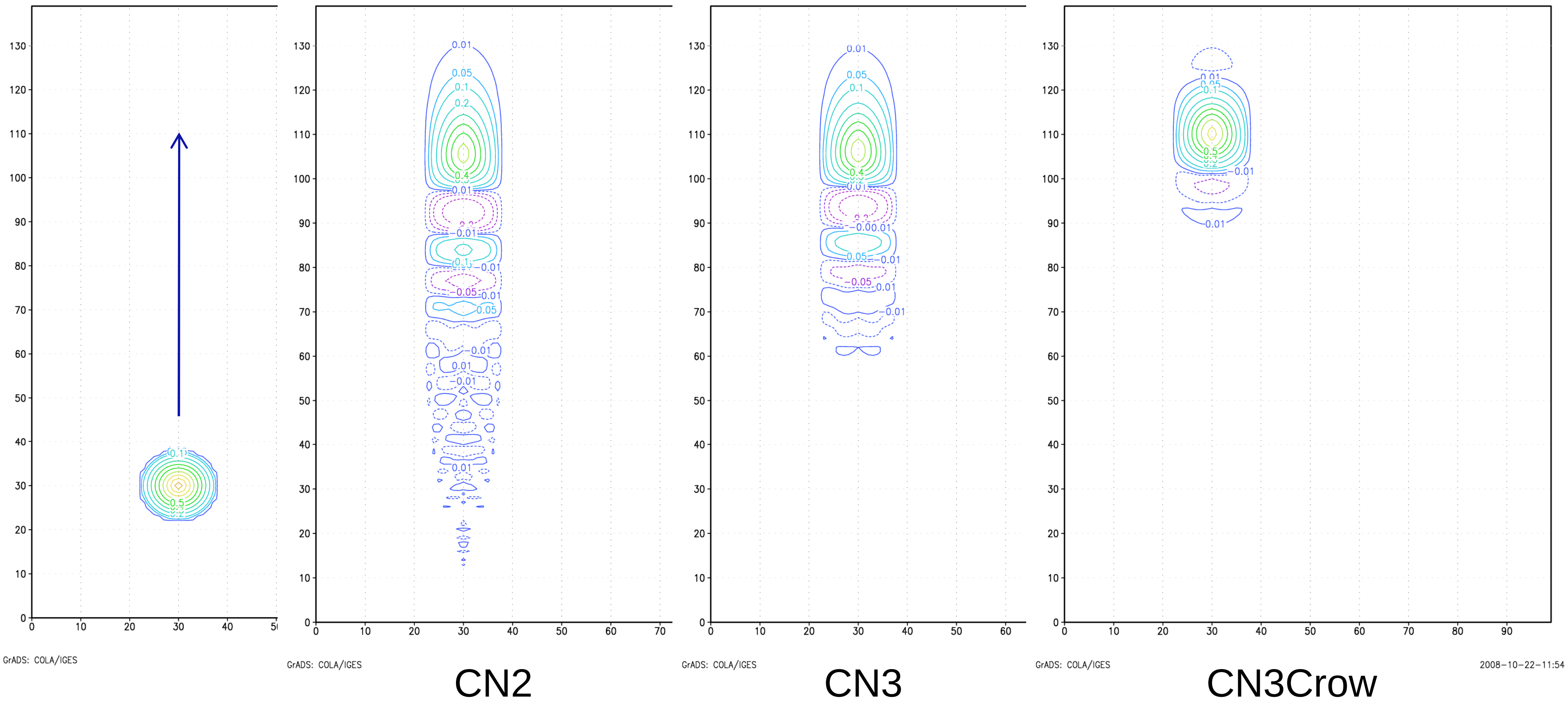
Pure advection test:
Only implicit scheme (i.e. u=0), w=2, dt=0.25 → C_z=0.5



Test 20081010_ay

Pure advection test:

Only implicit scheme (i.e. u=0), w=2, dt=1.5 → C_z=3



Test 20081010_ay

The current 3-stage RK-scheme in the COSMO-model

Wicker, Skamarock (2002) MWR

$$\text{solve the implicit scheme:} \quad \frac{\tilde{\phi} - \phi^n}{\frac{\Delta t}{3}} = \beta A_z(\tilde{\phi}) + (1-\beta) A_z(\phi^n) + A_x(\phi^n) + P(\phi^n) \qquad (1)$$

$$\text{... and define its tendency:} \quad L(\phi^n) := \frac{\tilde{\phi} - \phi^n}{\frac{\Delta t}{3}} \qquad (2)$$

$$\text{1. RK-substep:} \quad \phi^* = \phi^n + \frac{\Delta t}{3} L(\phi^n) \qquad (3)$$

$$\text{fast waves with tendency} \quad \frac{\phi^* - \phi^n}{\Delta t/3}, \text{ starting at } \phi^n \Rightarrow \phi^* \qquad (4)$$

$$\text{solve:} \quad \frac{\tilde{\phi} - [\alpha \phi^n + (1-\alpha)\phi^*]}{\frac{\Delta t}{2}} = \beta A_z(\tilde{\phi}) + (1-\beta) A_z(\phi^*) + A_x(\phi^*) + P(\phi^n) \qquad (5)$$

$$\text{... and define its tendency:} \quad L(\phi^*) := \text{lhs. of the above expression}^1 \qquad (6)$$

$$\text{2. RK-substep:} \quad \phi^{**} = \phi^n + \frac{\Delta t}{2} L(\phi^*) \qquad (7)$$

$$\text{fast waves with tendency} \quad \frac{\phi^{**} - \phi^n}{\Delta t/2}, \text{ starting at } \phi^n \Rightarrow \phi^{**} \qquad (8)$$

$$\text{solve:} \quad \frac{\tilde{\phi} - [\alpha \phi^n + (1-\alpha)\phi^{**}]}{\Delta t} = \beta A_z(\tilde{\phi}) + (1-\beta) A_z(\phi^{**}) + A_x(\phi^{**}) + P(\phi^n) \qquad (9)$$

$$\text{... and define its tendency:} \quad L(\phi^{**}) := \text{lhs. of the above expression} \qquad (10)$$

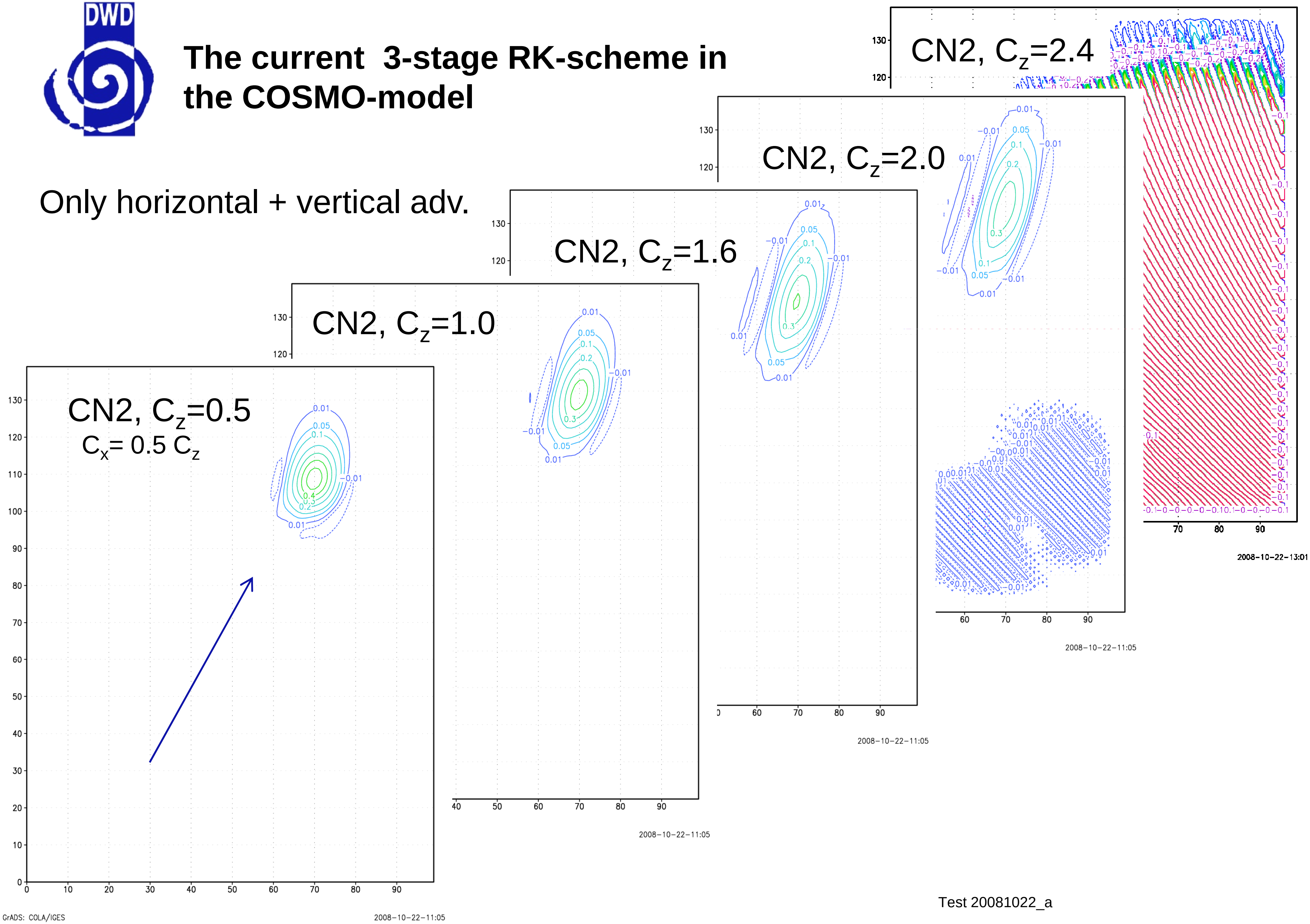
$$\text{3. RK-substep:} \quad \phi^{n+1} = \phi^n + \Delta t \, L(\phi^{**}) \qquad (11)$$

$$\text{fast waves with tendency} \quad \frac{\phi^{n+1} - \phi^n}{\Delta t}, \text{ starting at } \phi^n \Rightarrow \phi^{n+1} \qquad (12)$$

In the following: A_x = upwind 5th order

The current 3-stage RK-scheme in the COSMO-model

Only horizontal + vertical adv.



Test 20081022_a

Conclusions:

- The current scheme shows a certain damping.
  This seems to be typically for implicit schemes used inside of a Runge-Kutta-scheme (for the 'slow' part) (*Baldauf (2009), submitted to JCP*)
- The vertical advection is not unconditionally stable,
  despite the fact, that an implicit scheme is used
  (but up to now this was not a problem in all COSMO-DE or –EU runs)

Remark: for α=0 (i.e. without an overdamping):

- Bigger dispersion error
- Instability sets in much earlier (at dt=0.8 → C_z=1.6)

New proposal: complete operator splitting

$$\frac{\tilde{\phi} - \phi^n}{\Delta t} = R_x(\phi^n)$$

$$\frac{\phi^{n+1} - \tilde{\phi}}{\Delta t} = \beta A_z(\phi^{n+1}) + (1-\beta) A_z(\tilde{\phi})$$

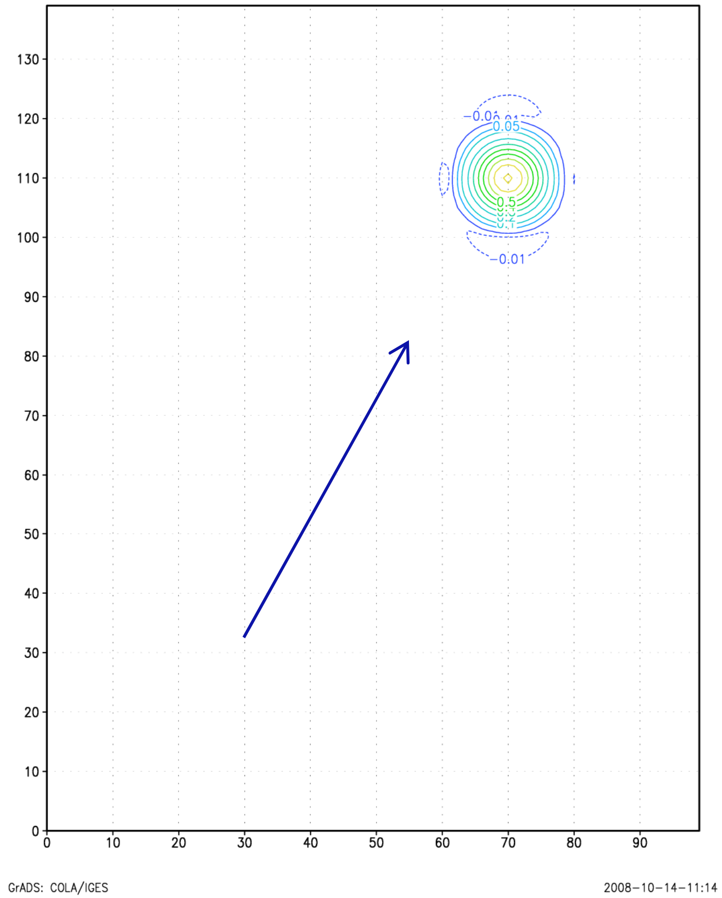
R_x = complete RK3-scheme, but without A_z

advantages:

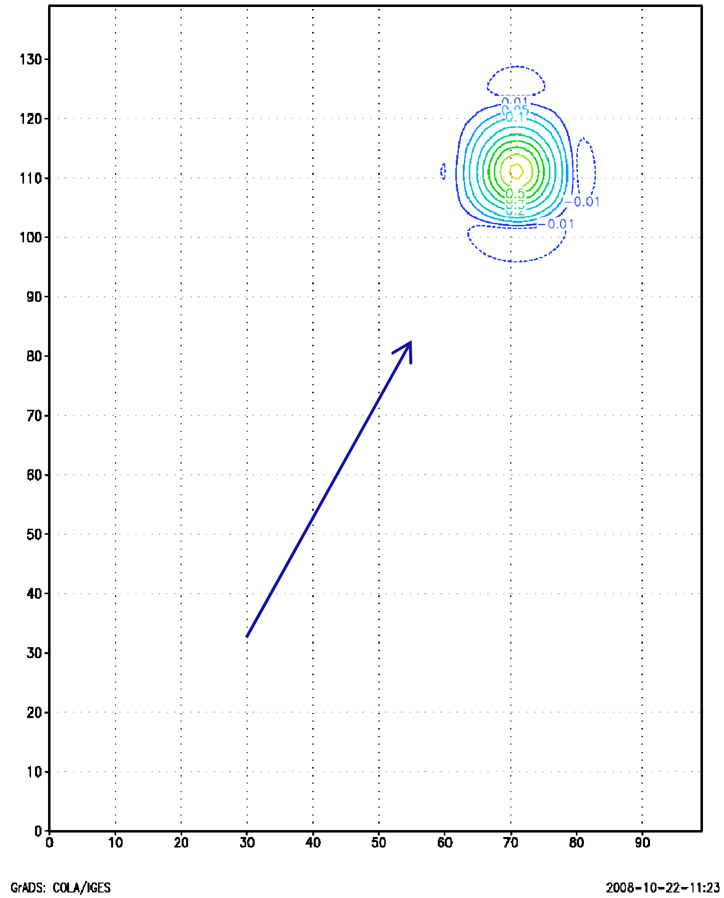
- no implicit scheme occurs inside of the RK
- if the numerical operators commute, then the stability properties of the single operators are passed on to the whole scheme (*LeVeque and Oliger, 1983*)
- ‚overdamping' is not needed (→ α=0 is possible)
- ‚expensive' vertical advection is called only once / timestep

Complete operator splitting, CN3Crow

dt=0.25 → C_z=0.5



dt=1.2 → C_z=2.4



Test 20081010_a

Question: does the 'complete operator splitting' work together with fast (sound-, gravity-) waves? (splitting-error, stability, ...)

Linearised shallow water equations

$$\begin{aligned}
\frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + V_0 \frac{\partial u}{\partial y} &= -g \frac{\partial h}{\partial x} + M_x \\
\frac{\partial v}{\partial t} + U_0 \frac{\partial v}{\partial x} + V_0 \frac{\partial v}{\partial y} &= -g \frac{\partial h}{\partial y} + M_y \\
\frac{\partial h}{\partial t} + U_0 \frac{\partial h}{\partial x} + V_0 \frac{\partial h}{\partial y} &= -H_0 \, D \\
\text{with } D &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}
\end{aligned}$$

Divergence damping:

$$\begin{aligned}
M_x &= \alpha_{Div} \frac{\partial D}{\partial x} \\
M_y &= \alpha_{Div} \frac{\partial D}{\partial y}
\end{aligned}$$

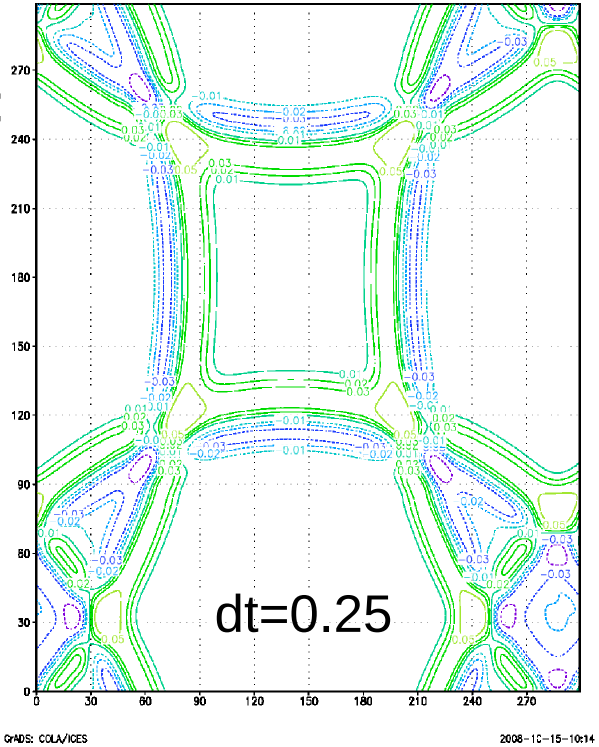
Advective transport with velocity (U_0, V_0)

fast wave expansion with velocity $c_{wav} = \sqrt{g H_0}$

Here: apply the new implicit advection to the y-direction

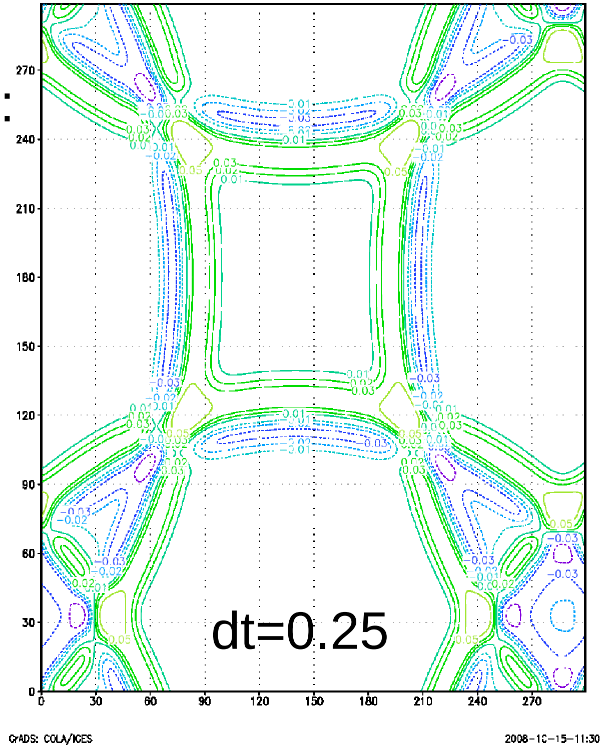
For reference:

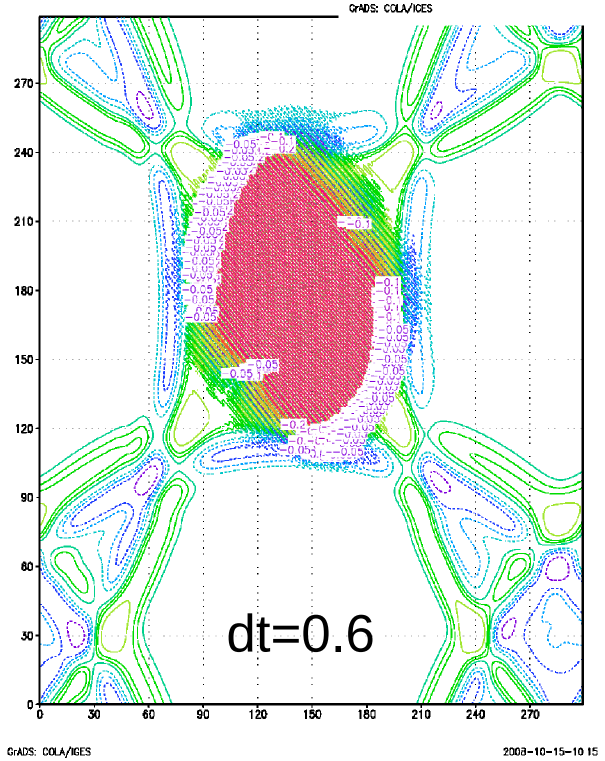
upwind5
in y-direction



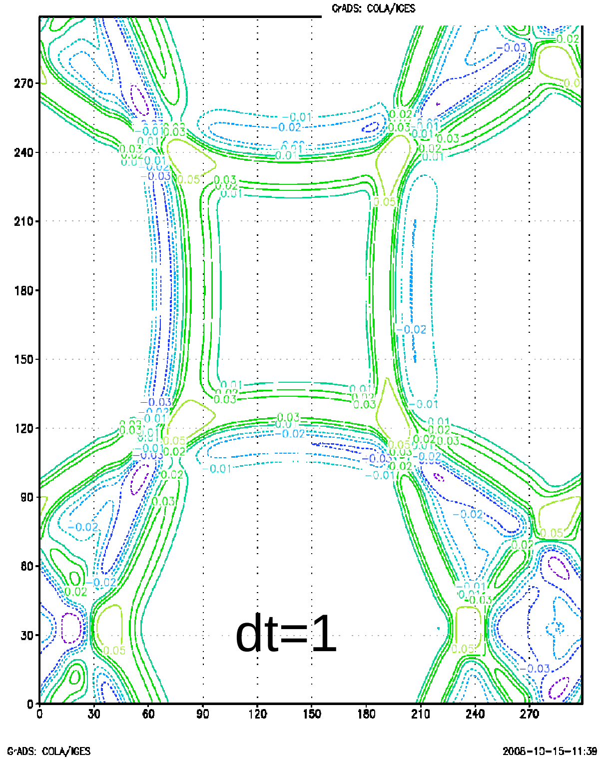
New scheme:

CN3Crow
in y-direction





Height field



Do better combinations exist between horizontal and vertical advection?

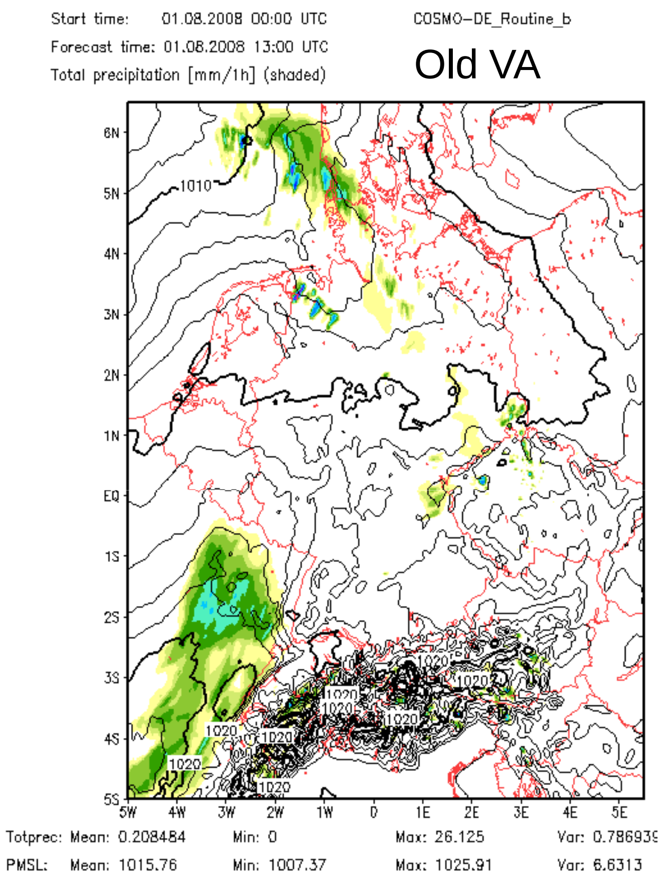
- complete operator splitting, but with horizontal advection as 'rhs':

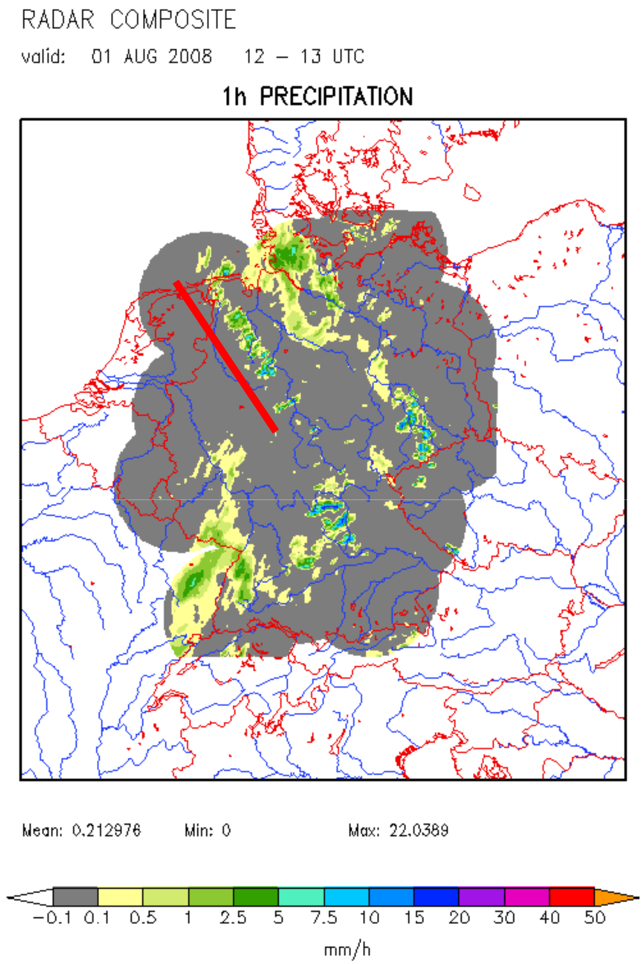
$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \beta A_z(\phi^{n+1}) + (1-\beta) A_z(\phi^n) + R_x(\phi^n)$$

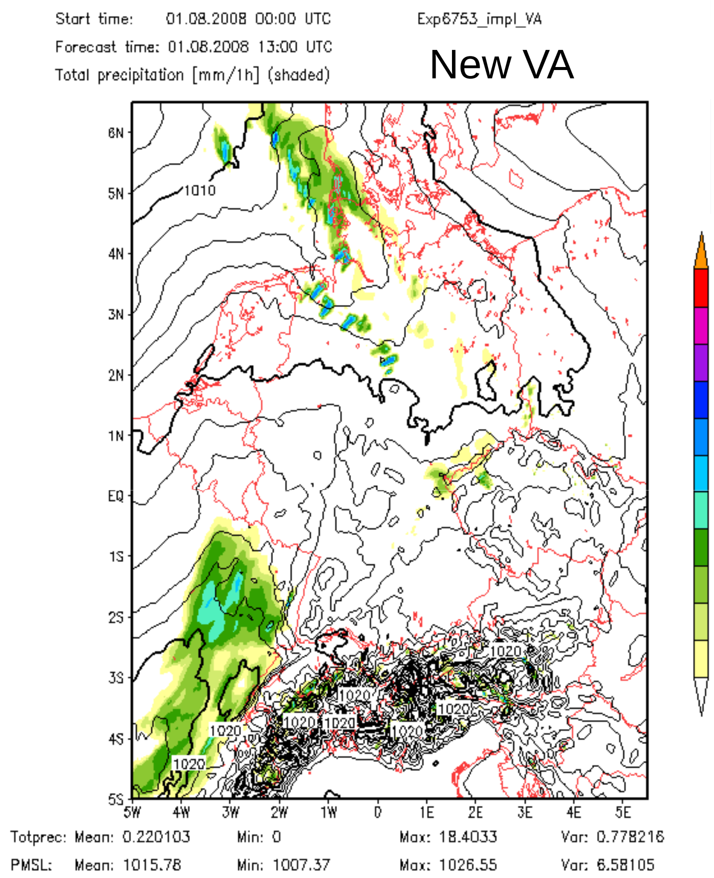
  → adv. test unstable!

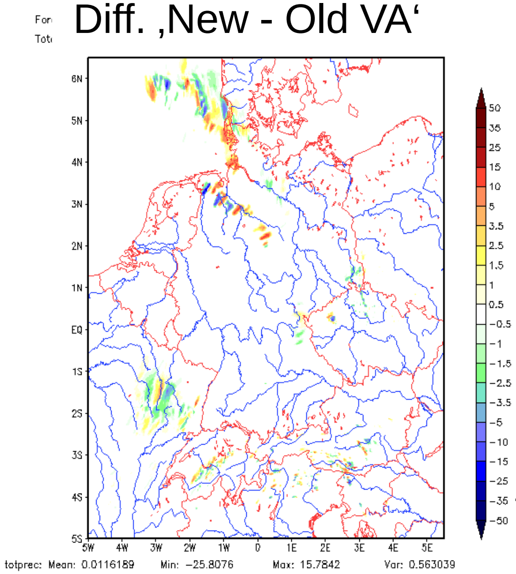
  It seems generally not to be a good idea to use explicit parts as a rhs of an implicit scheme
- tendencies of the vertical advection analogous to the physical tendencies
  → adv. test unstable
- implicit scheme only in 3rd RK-sub step → adv. test unstable
- mixing: explicit in RK 1st+2nd sub step, implicit in 3rd RK-sub step
  → adv. test: strongly damping or unstable
- 'partial operator splitting': → adv.tests ok., but shallow water equations unstable

$$\begin{aligned}
\phi^{*} &= \phi^n + \frac{\Delta t}{3} A_x(\phi^n) \\
\phi^{**} &= \phi^n + \frac{\Delta t}{2} A_x(\phi^*) \\
\phi^{***} &= \phi^n + \Delta t\, A_x(\phi^{**}) \\
\phi^{n+1} &= \phi^{***} + \Delta t \left[\beta A_z(\phi^{n+1}) + (1-\beta) A_z(\phi^{***})\right]
\end{aligned}$$



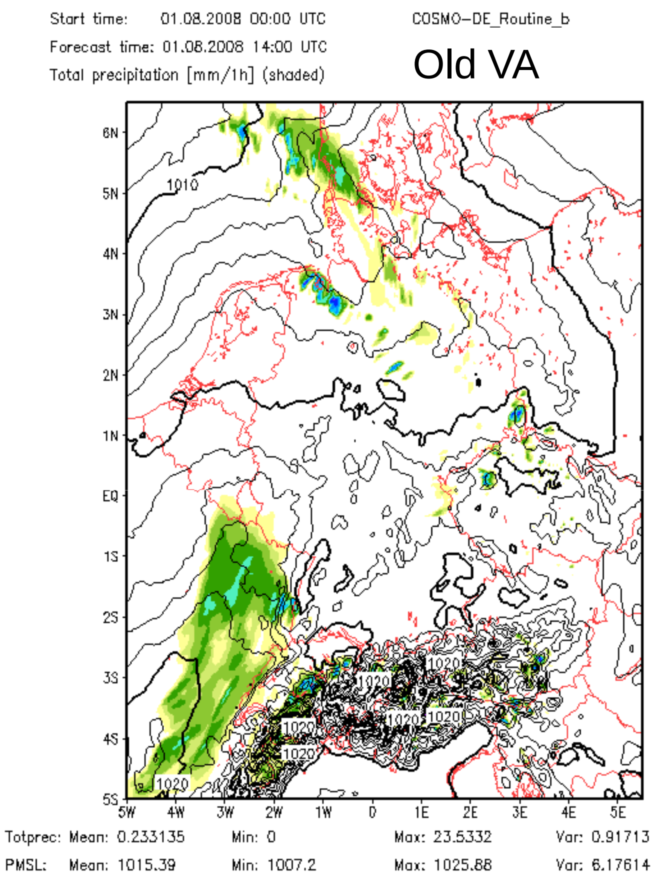


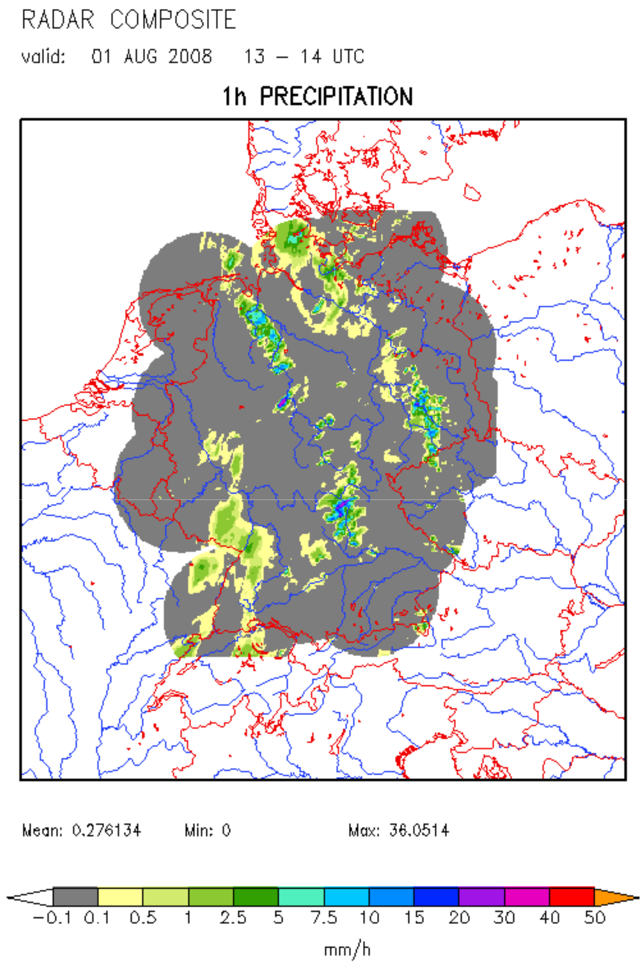


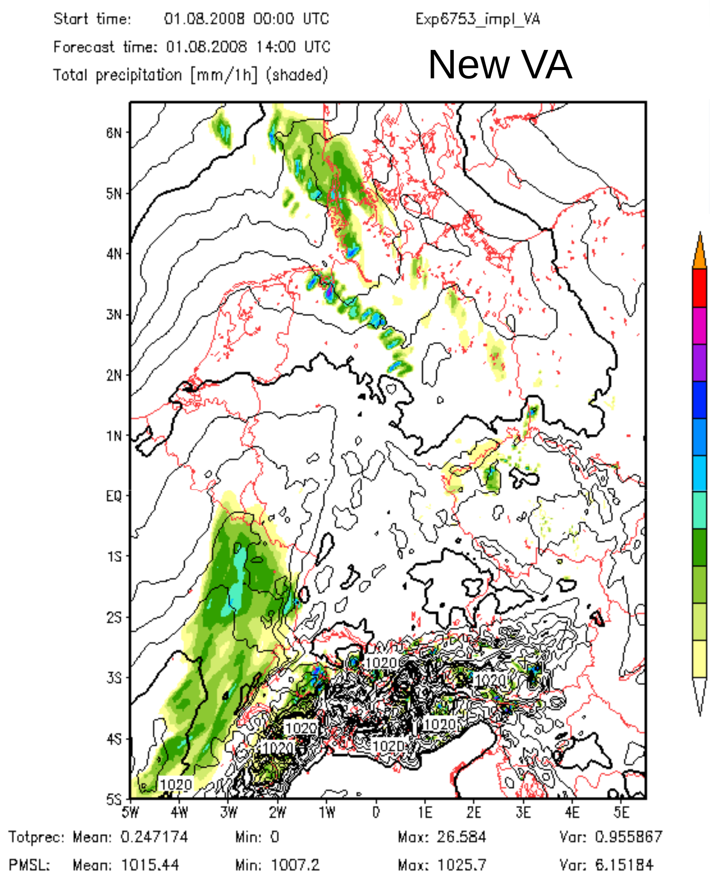


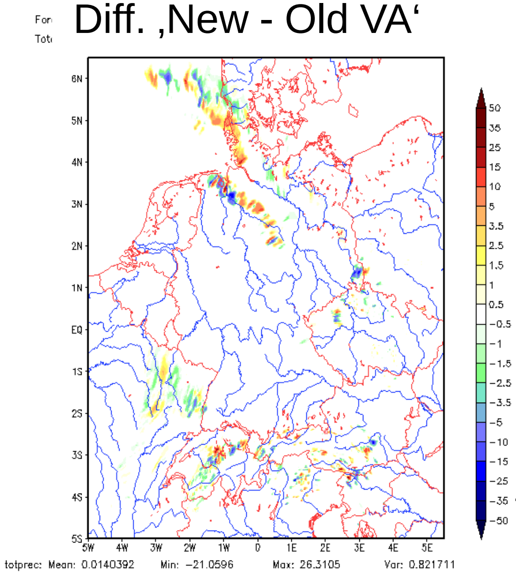
Real case study:
COSMO-DE (2.8 km resolution) for the ‚01.08.2008‘, 0 UTC run
1h-precipitation sum at 13 UTC

‘New VA’ = ‘Complete operator splitting’ with ‘CN3Crow’

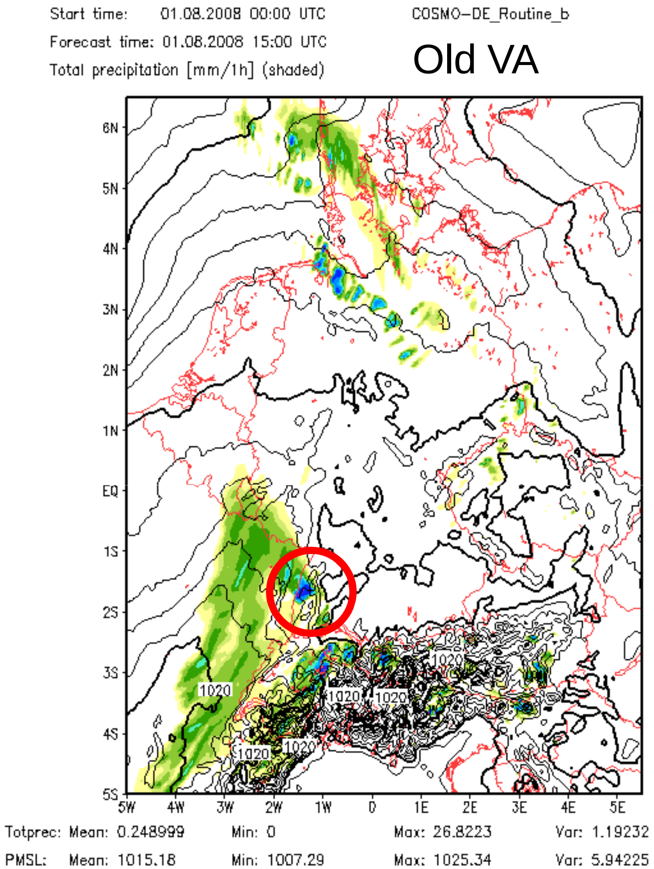


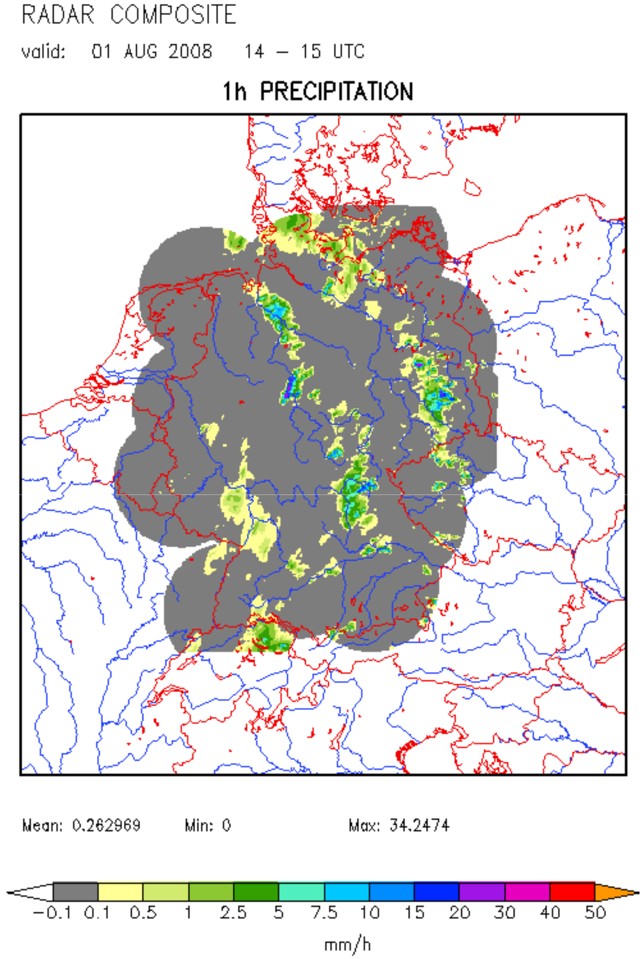


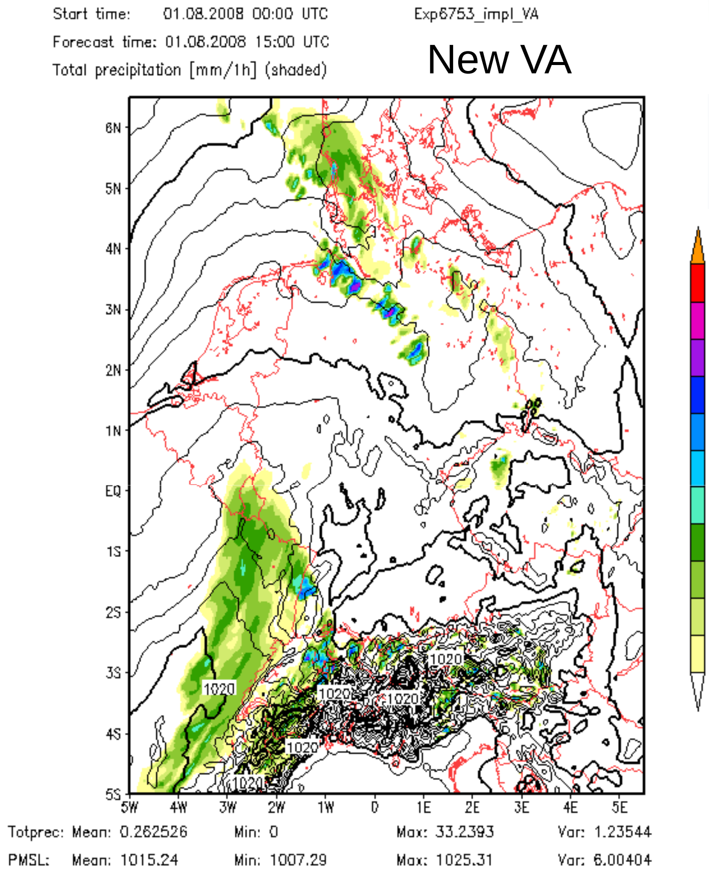


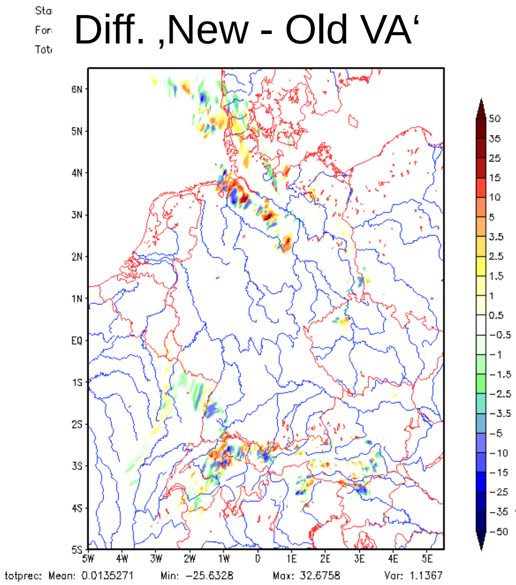


Real case study:
COSMO-DE (2.8 km resolution) for the ‚01.08.2008‘, 0 UTC run
1h-precipitation sum at 14 UTC



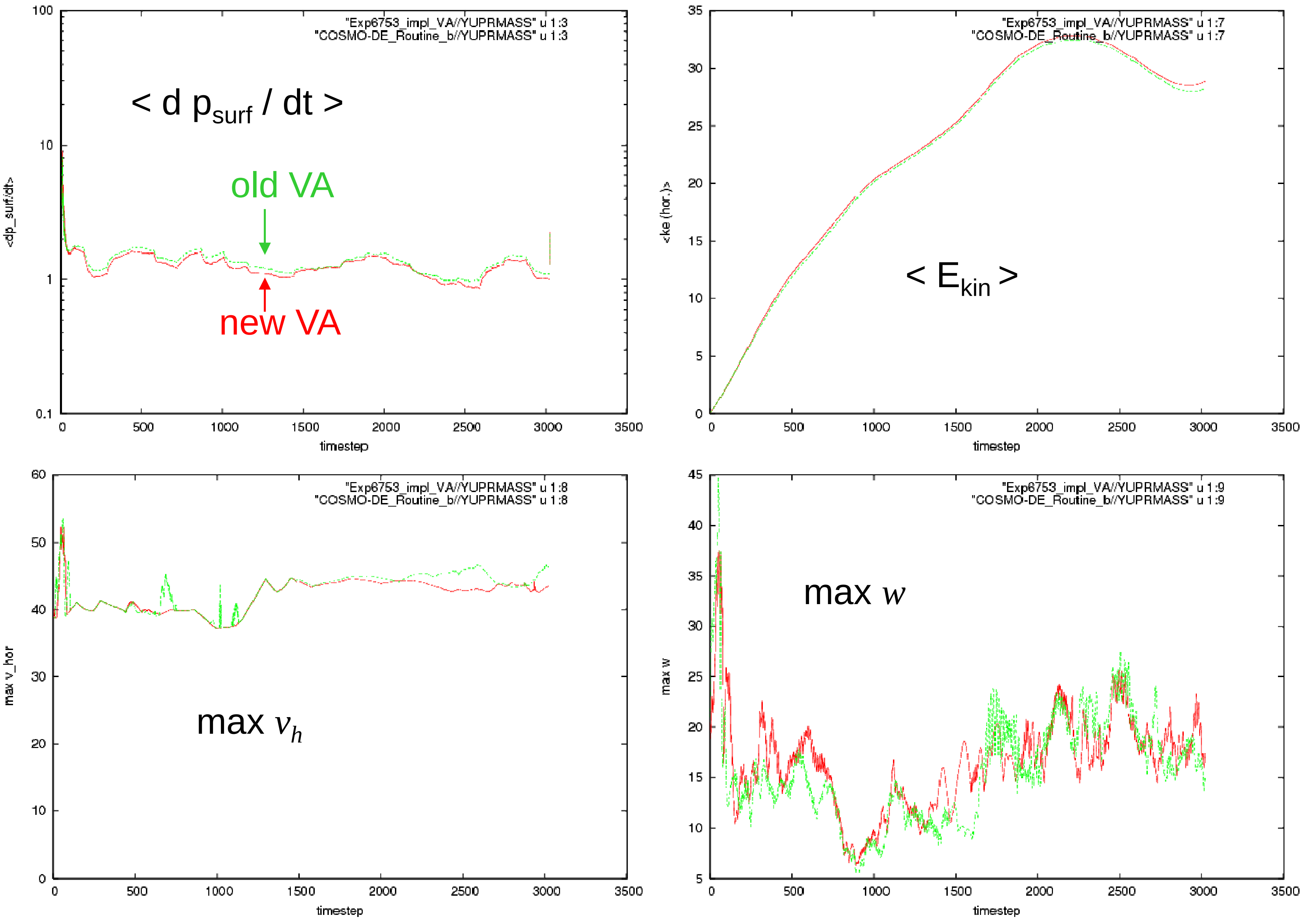




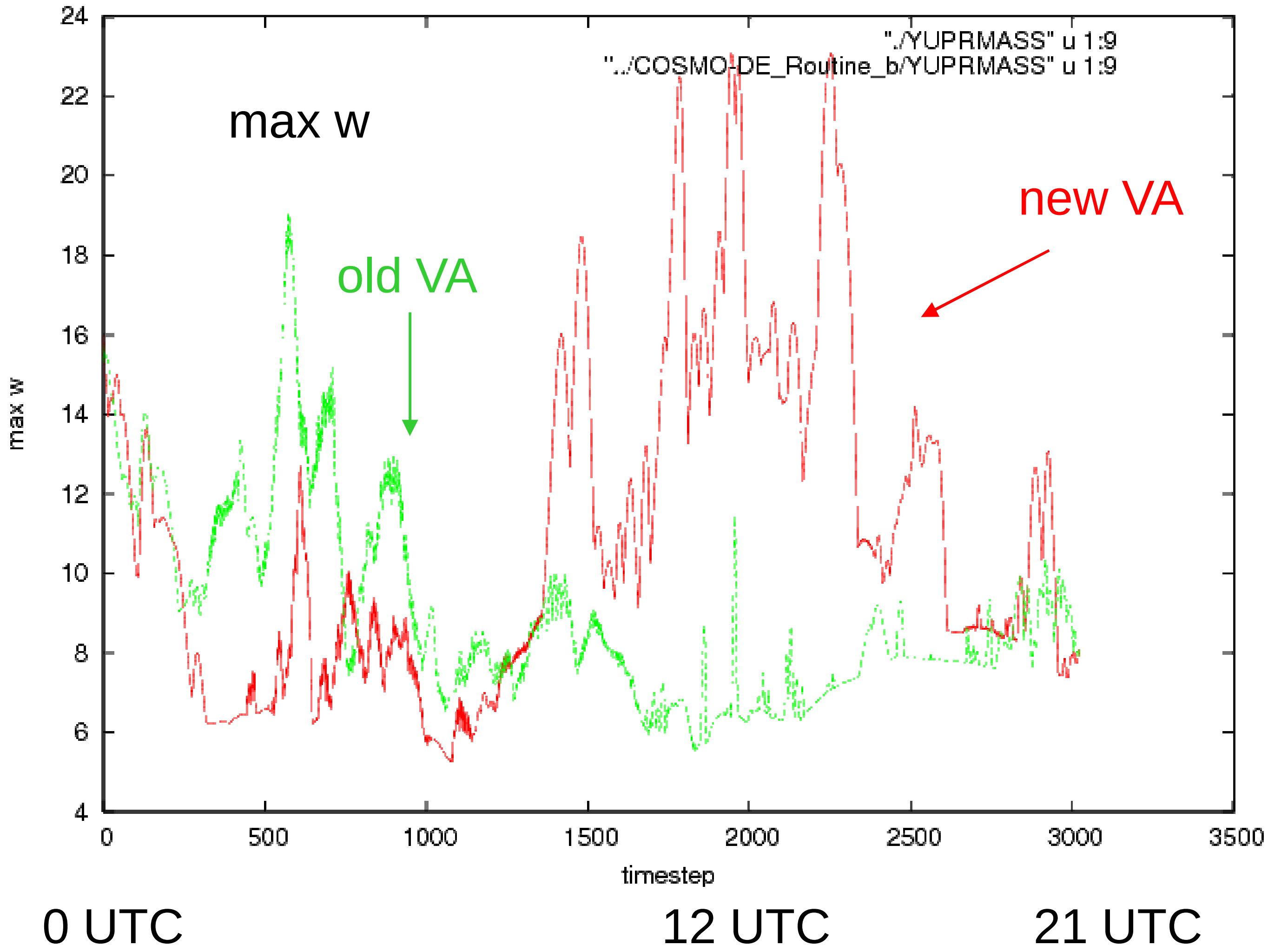


Real case study:
COSMO-DE (2.8 km resolution) for the ‚01.08.2008‘, 0 UTC run
1h-precipitation sum at 15 UTC

01.08.2008, 0 UTC, mean/max values



example: 16.08.2008, 0 UTC - run

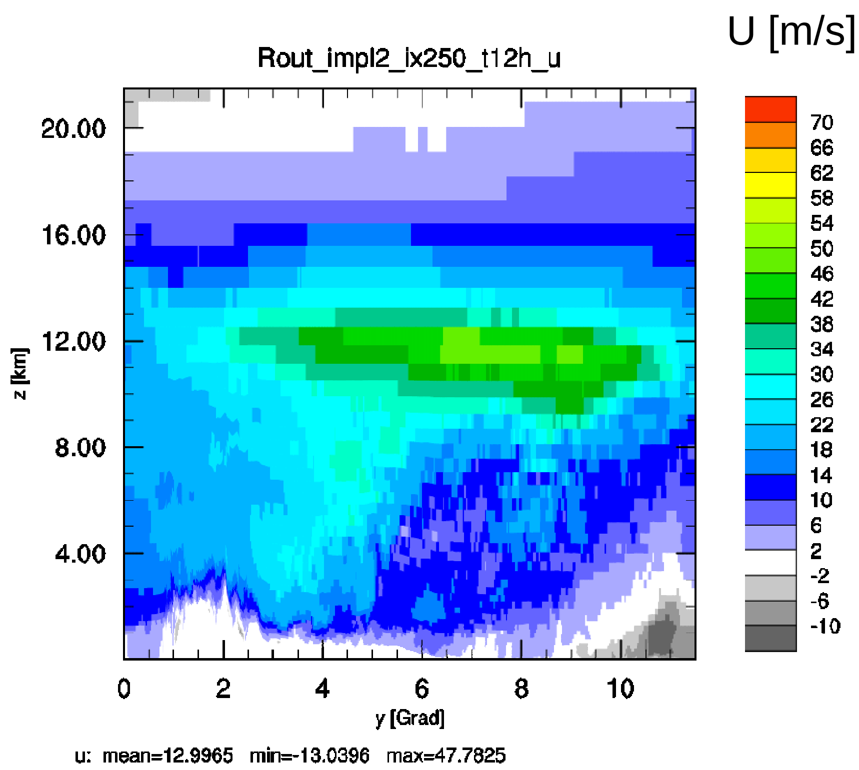


The new VA somtimes produces higher vertical velocities at noon

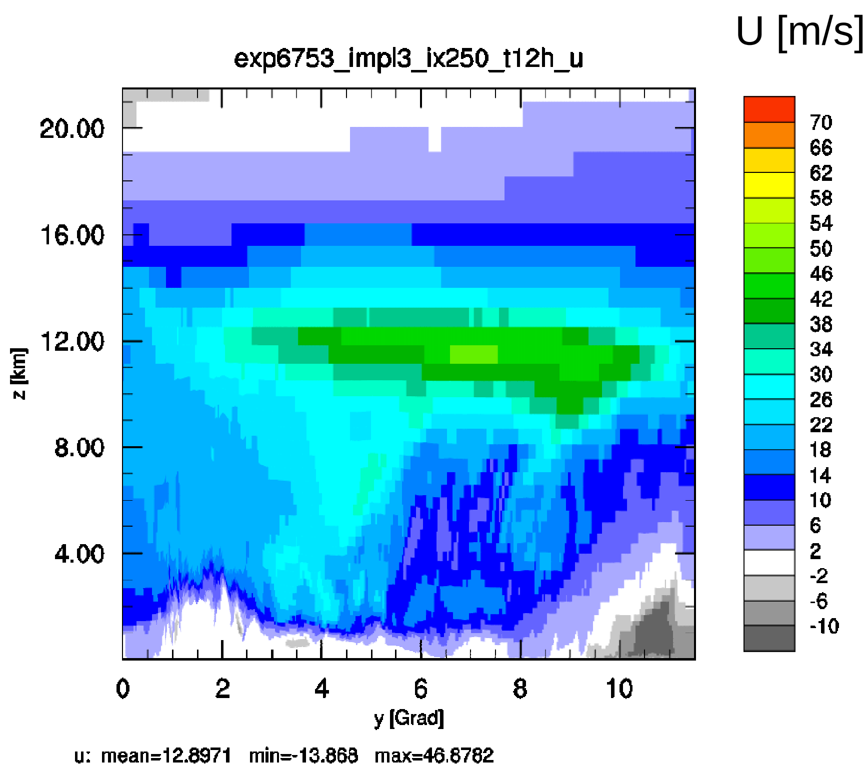
Vertical cross section of zonal velocity

'12.08.2008 0 UTC run' after 12 h

Old VA



New VA



slightly less noisy velocity field

Summary

The current implicit vertical advection scheme possess a relatively strong damping and is formally not unconditionally stable.

From all of the tested alternatives only the 'complete operator splitting' (= vertical advection outside of the RK-scheme) with CN3 or CN3Crow has proven to be superior:

- improved advection properties in idealized advection tests
- unconditionally stable in C_z
- works also in combination with fast waves
- plausible results in idealized and real cases
- computational amount is only slightly increased
- runs stable for COSMO-DE (2.8 km) simulations during a summer period ('28.07.-21.08.2008')
  (an instability at '12.08.' could be cured by an additional calc. of the contravariant vertical velocity)
- stable calculation of a winter time storm event ('10.02.2009')

Outlook

- Synoptic verification
- Correct simulation of inversion layers?

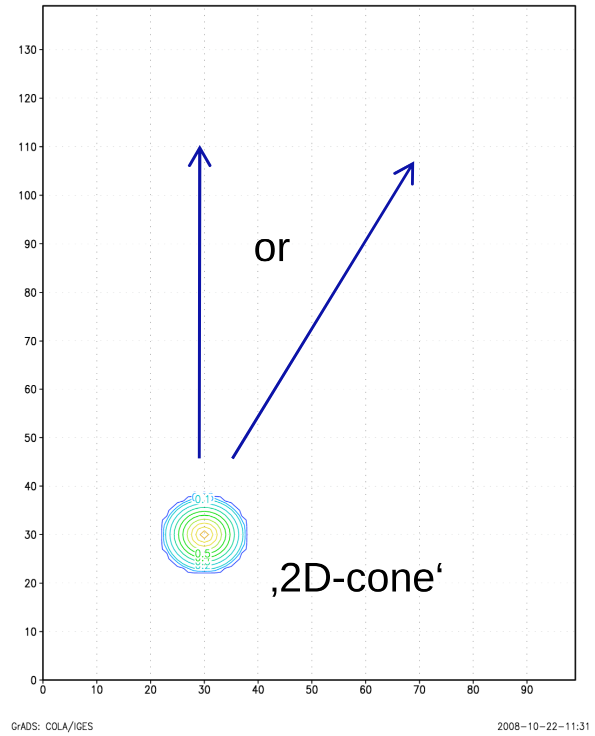
Demonstrate the ability of the implicit schemes: 2D-advection tests

In the following:

- the 3-stage Runge-Kutta is used as time integration scheme
- the horizontal advection always is a 5th order upwind scheme

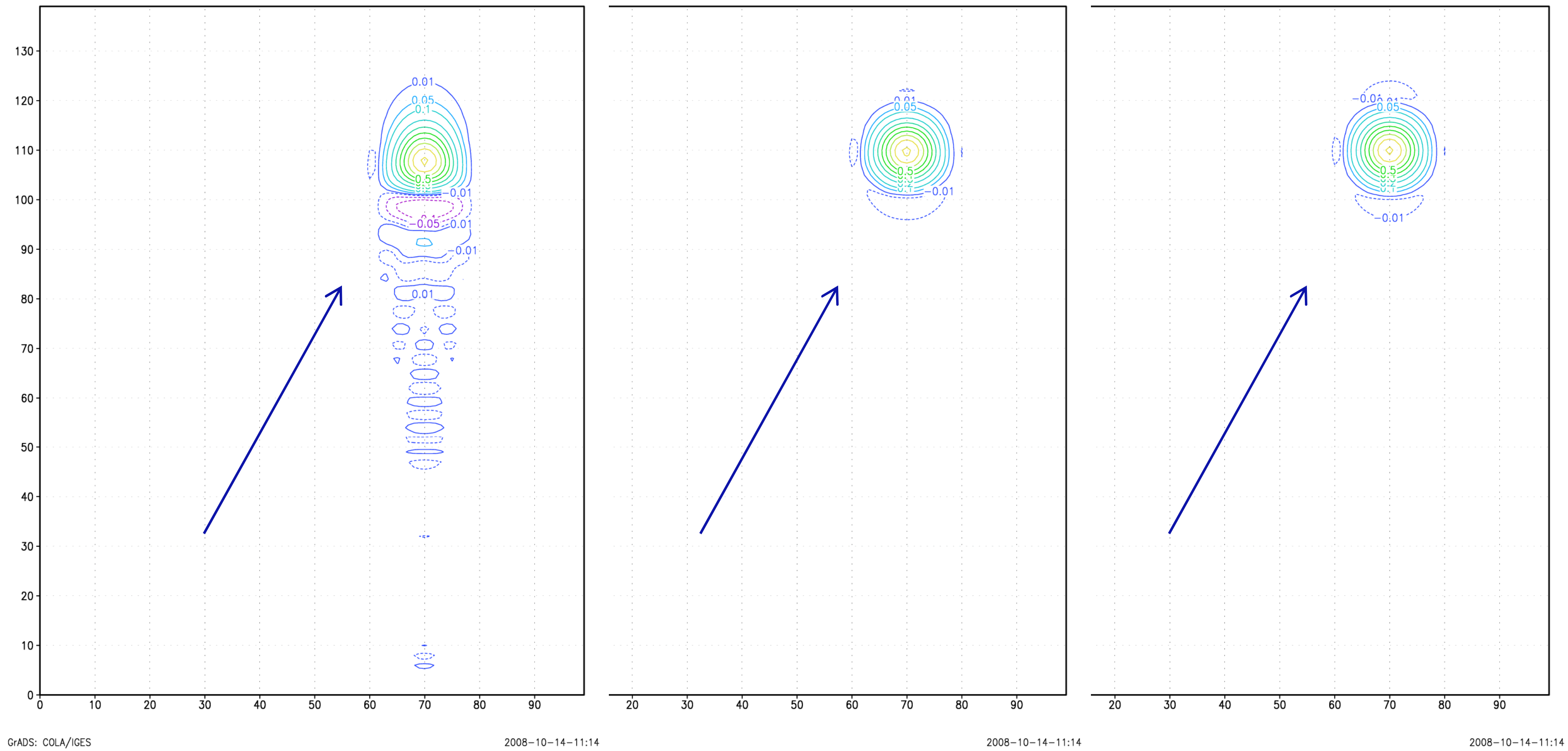
Wicker, Skamarock, 2002, MWR

Initial state:



Complete operator splitting, dt=0.25 → C_z=0.5

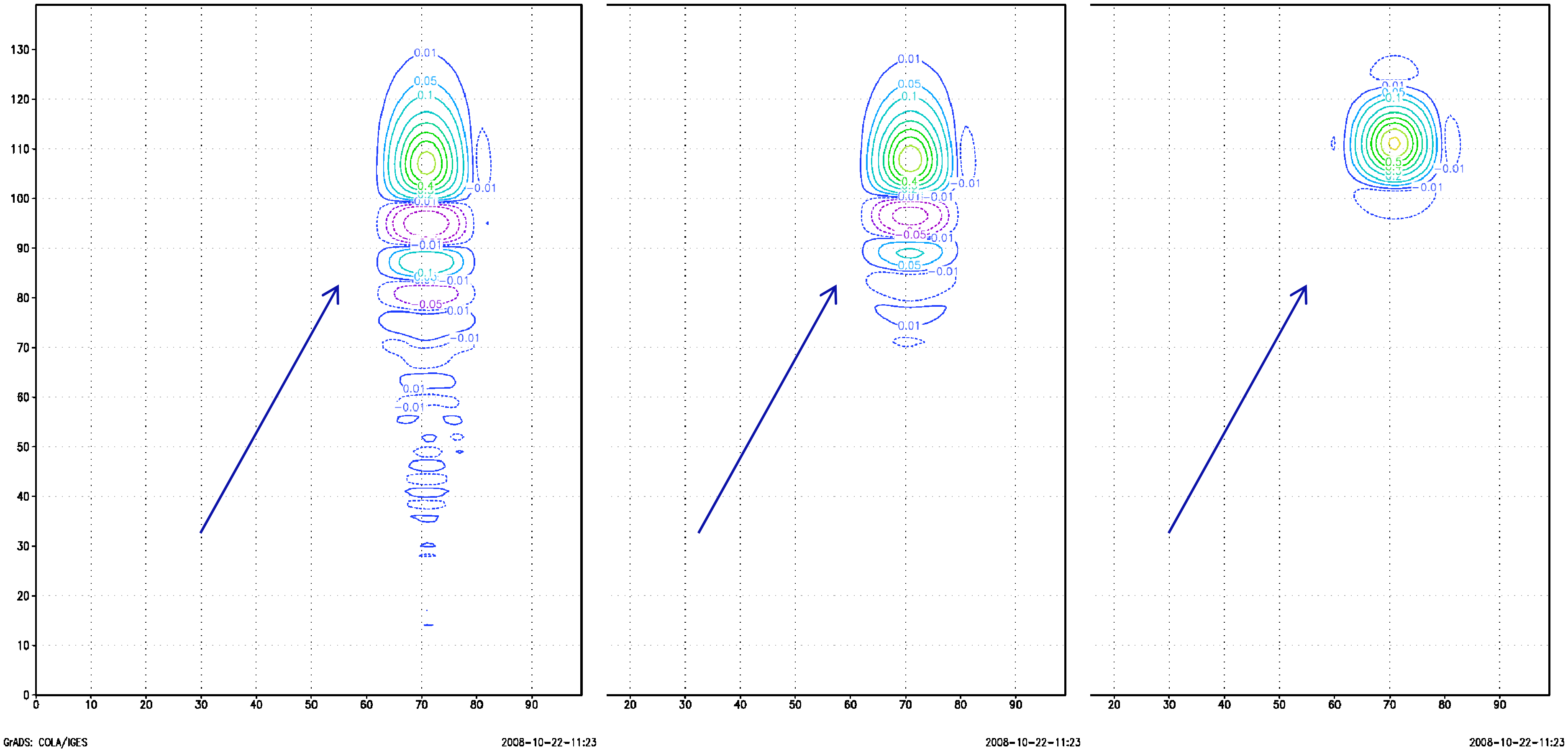
CN2 | CN3 | **CN3Crow**



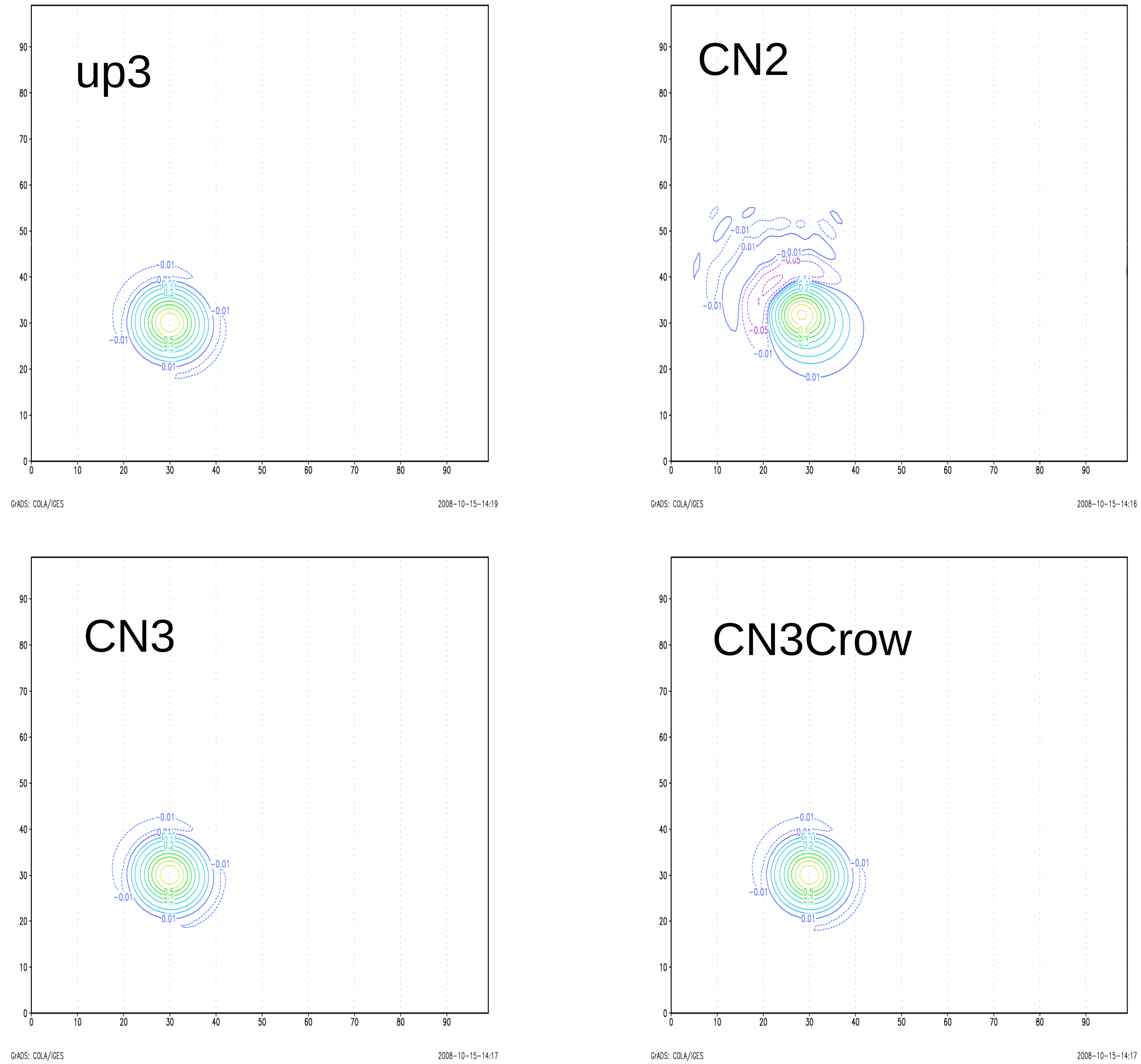
Test 20081010_a

Complete operator splitting, dt=1.2 → C_z=2.4

CN2 CN3 **CN3Crow**



Test ‚solid body rotation' (1 turn around)

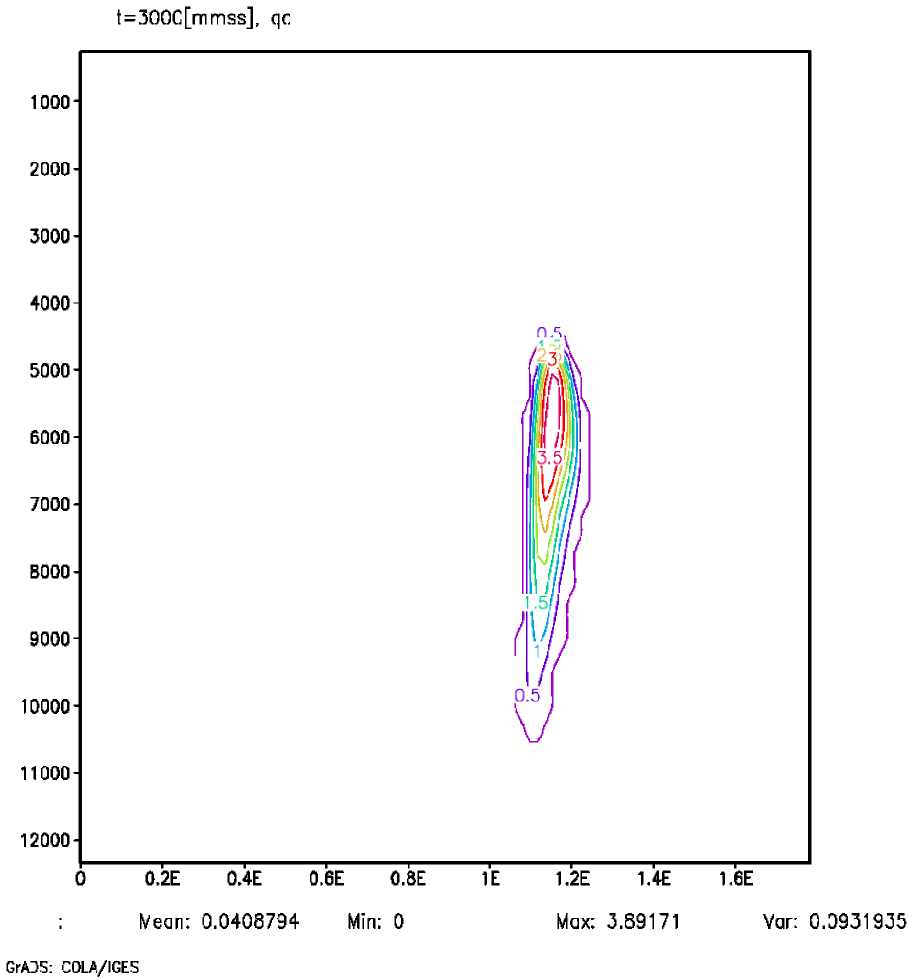


COSMO-model:

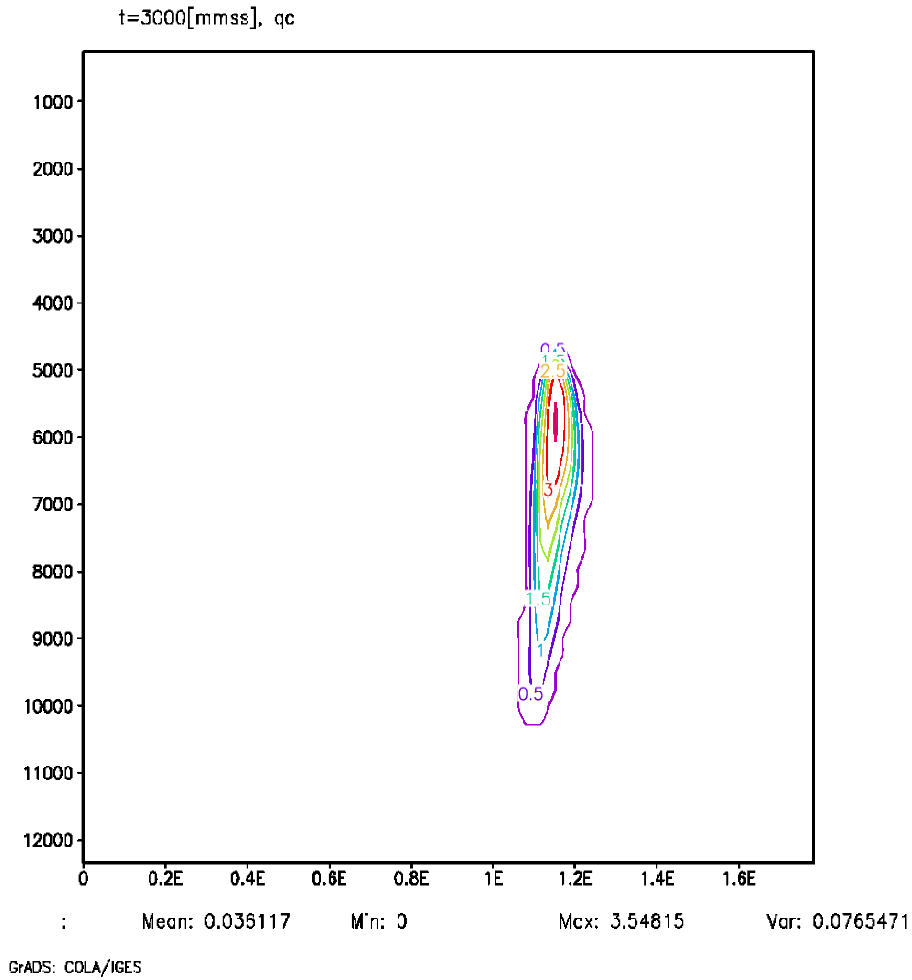
behaviour with complete dynamical core (RK3) + simplified physics (only cond./evap

Weisman, Klemp (1982)-test case

q_c, after 30 min.



CN2 /alt



CN3 / kompl. Oper-splitt.

Efficiency:

CN2 / current scheme:
solves a tridiagonal system: comp. effort ~ 3 N
call in every RK-substep = 3 times / timestep

CN3, ... / complete operator splitting:
solves a pentadiagonal system: comp. effort ~ 13 N
call 1 times / timestep

uses 'Numerical recipes' routines bandec, banbks, which where optimized for vector computers (NEC SX-8R)

gprof: CN2 / bisheriges Schema

```
ngranularity: Each sample hit covers 4 bytes. Time: 10324.90 seconds

                                  called/total       parents
index  %time    self descendents  called+self    name        index
                                  called/total       children

              214.25      5765.10    3025/3025           .organize_dynamics [3]
[4]      57.9  214.25      5765.10    3025                .__src_runge_kutta_NMOD_org_runge_kutta [4]
             2612.57        35.15    9075/9075           fast_waves_runge_kutta [5]
               17.98       983.63    3025/3025           .__src_advection_rk_NMOD_advection_pd [8]
=====>        814.85         0.00    9075/9075           complete_tendencies_uvwtpp [10]
              234.86       305.48    9075/9075           .__src_advection_rk_NMOD_advection [12]
```

gprof: CN3Crow / komplettes Operatorsplitting

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ngranularity: Each sample hit covers 4 bytes. Time: 10248.44 seconds

               276.82     5682.12    3025/3025        .organize_dynamics [3]
[4]      58.1  276.82     5682.12    3025         .__src_runge_kutta_NMOD_org_runge_kutta [4]
              2337.45       36.04    9075/9075        .__fast_waves_rk_NMOD_fast_waves_runge_kutta
=====>         248.67      829.96    3025/3025        complete_tend_uvwtpp_cn3crow_nr [7]
                17.02      956.69    3025/3025        .__src_advection_rk_NMOD_advection_pd [9]
               223.75      305.84    9075/9075        .__src_advection_rk_NMOD_advection [13]
               216.53        0.00    3025/3025        implicit_vert_diffusion_uvwt [21]
                47.27      141.61    3025/3025        .__hori_diffusion_NMOD_comp_hori_diff [23]

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               248.67      829.96    3025/3025        .__src_runge_kutta_NMOD_org_runge_kutta [4]
[7]      10.5  248.67      829.96    3025         .complete_tend_uvwtpp_cn3crow_nr [7]
               306.41      523.53   15125/15125       .__numeric_utilities_NMOD_solve_5banddiag [1

-----------------------------------------------

               306.41      523.53   15125/15125       complete_tend_uvwtpp_cn3crow_nr [7]
[11]      8.1  306.41      523.53   15125         .__numeric_utilities_NMOD_solve_5banddiag [11]
               313.96        0.00 59486625/59486625     .__numeric_utilities_NMOD_bandec [18]
               209.46        0.00 59486625/59486625     .__numeric_utilities_NMOD_banbks [22]
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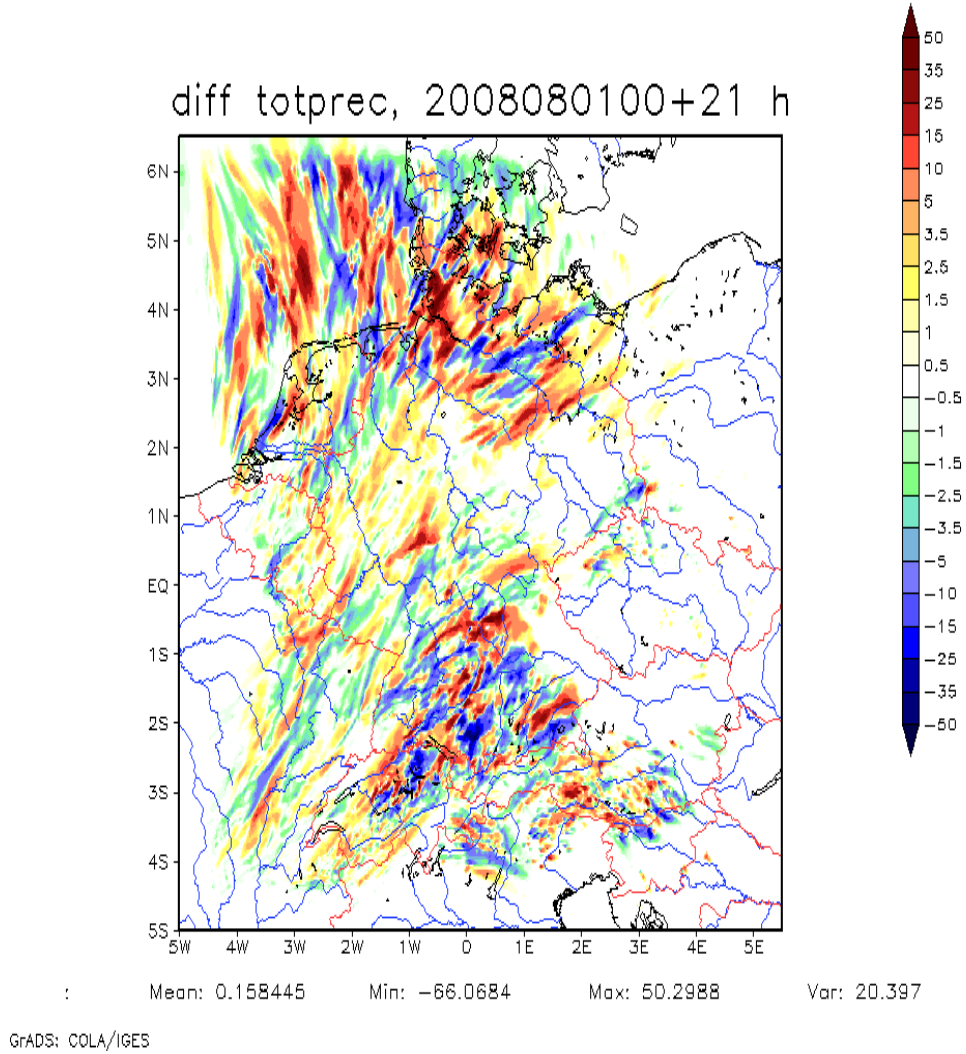
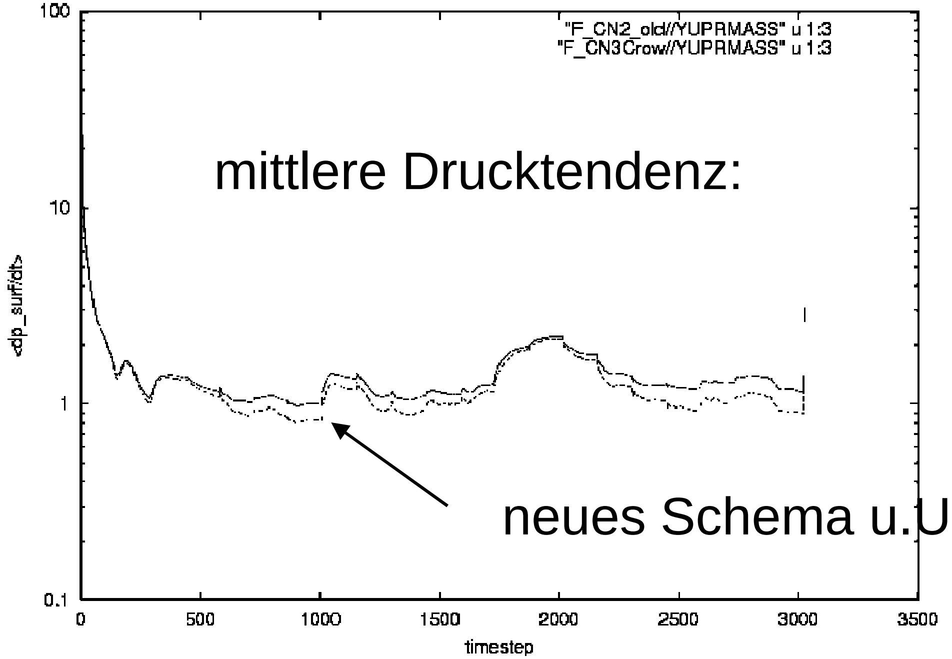


01.08.2008, 0 UTC
21 h precipitation sum
Diff. ‚New – Old. VA‘



Realer Testfall ('22.08.2008, 0 UTC')

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