High order conserving schemes on arbitrary polygonal grids

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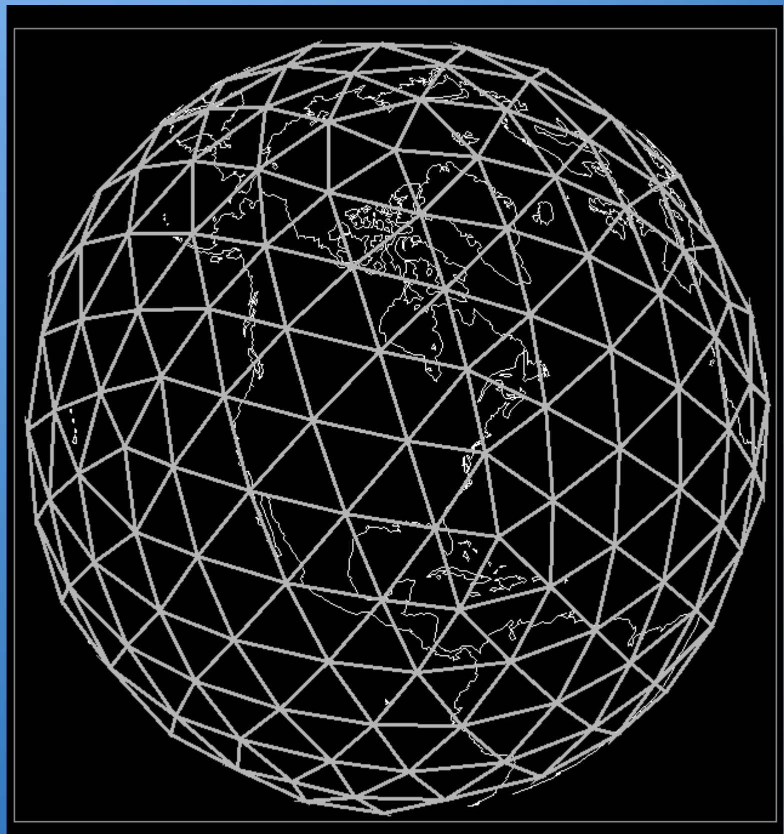
Conservation and high order

- ┬The Importance of Conservation
- The Importance of Maintaining Second or Third Order Accuracy:
  1. Bulls eyes at special points of the icosahedral grid
  2. Big problems with sudden grid changes in 1-d
     - Many models Use Grid Regularisation on the Icosahedron and a Smooth Vertical Grid Structure
     - Problems with Irregular Resolution can be Traced Back to First Order appoximations at Special Points
     - Schemes with Uniformly Second or Third Order Approximation Have no Bulls Eyes and are Robust

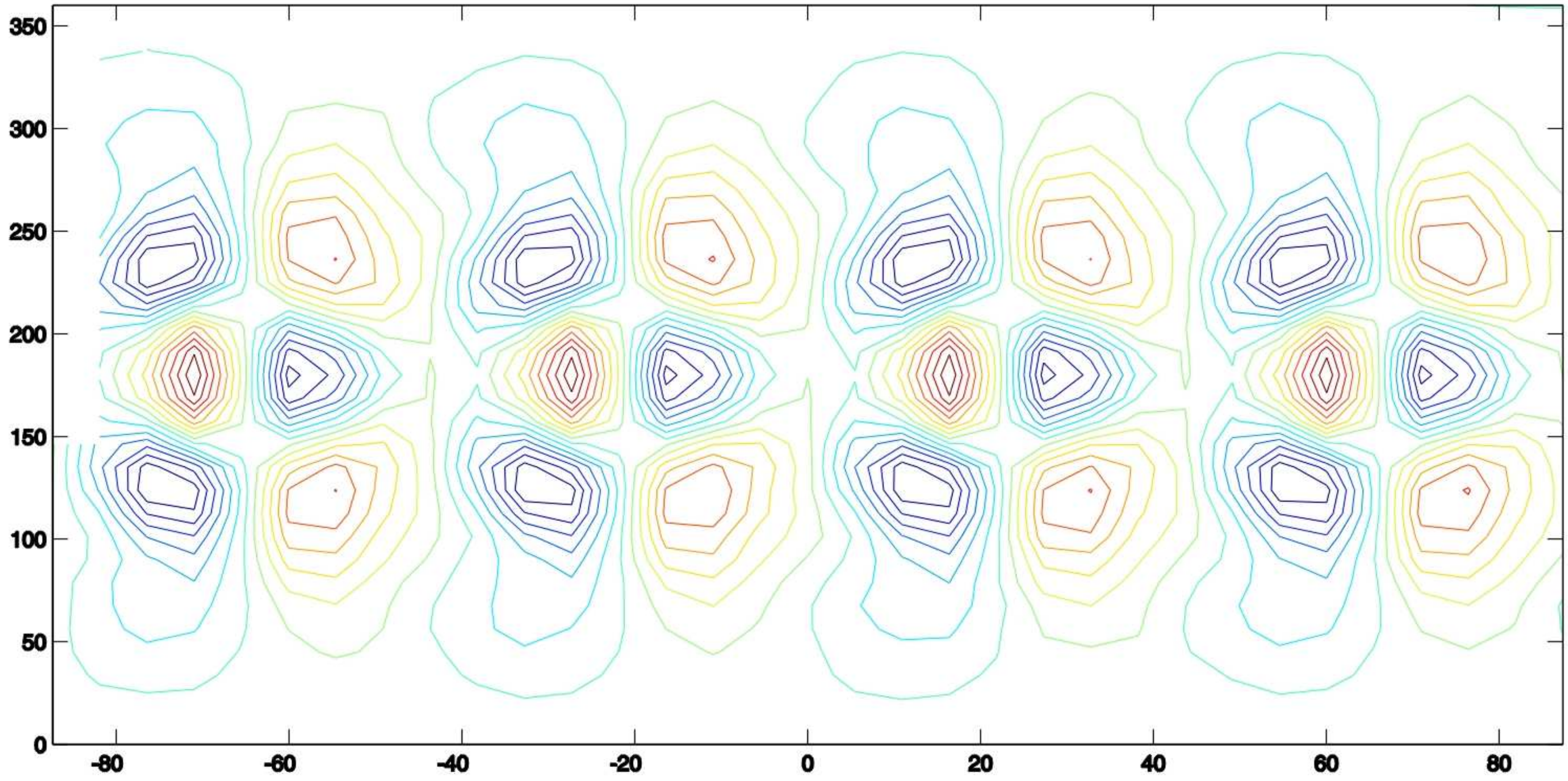
Icosahedral Grids

Rhomboidal Grid: hexagonal and pentagonal stencils and dual cells

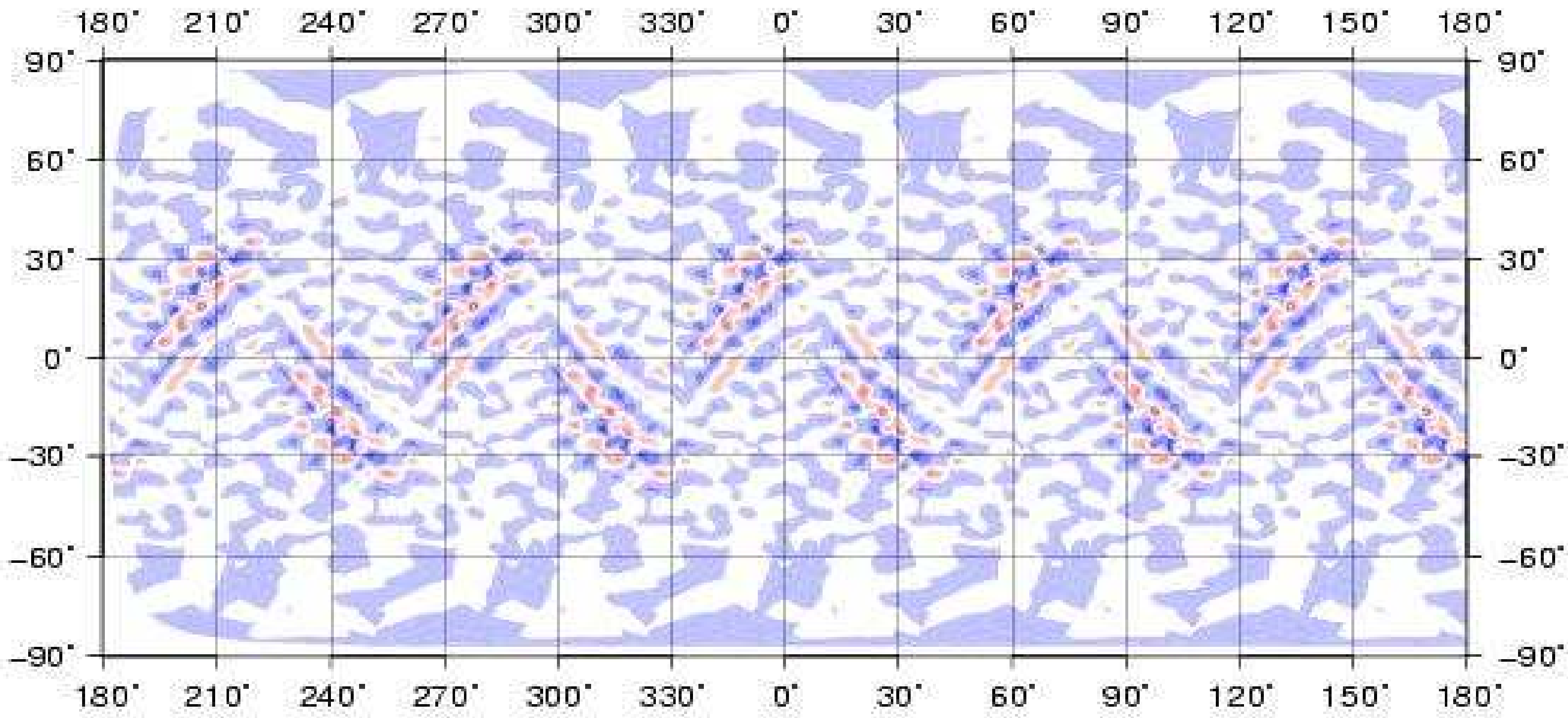
- An Uniformly Third Order Scheme was implemented and produces no Bulls Eyes (MWR **136** (2008) pp 2483,Steppeler and Ripodas)
- Are ***flux based conserving second or third order schemes*** possible ? (YES)



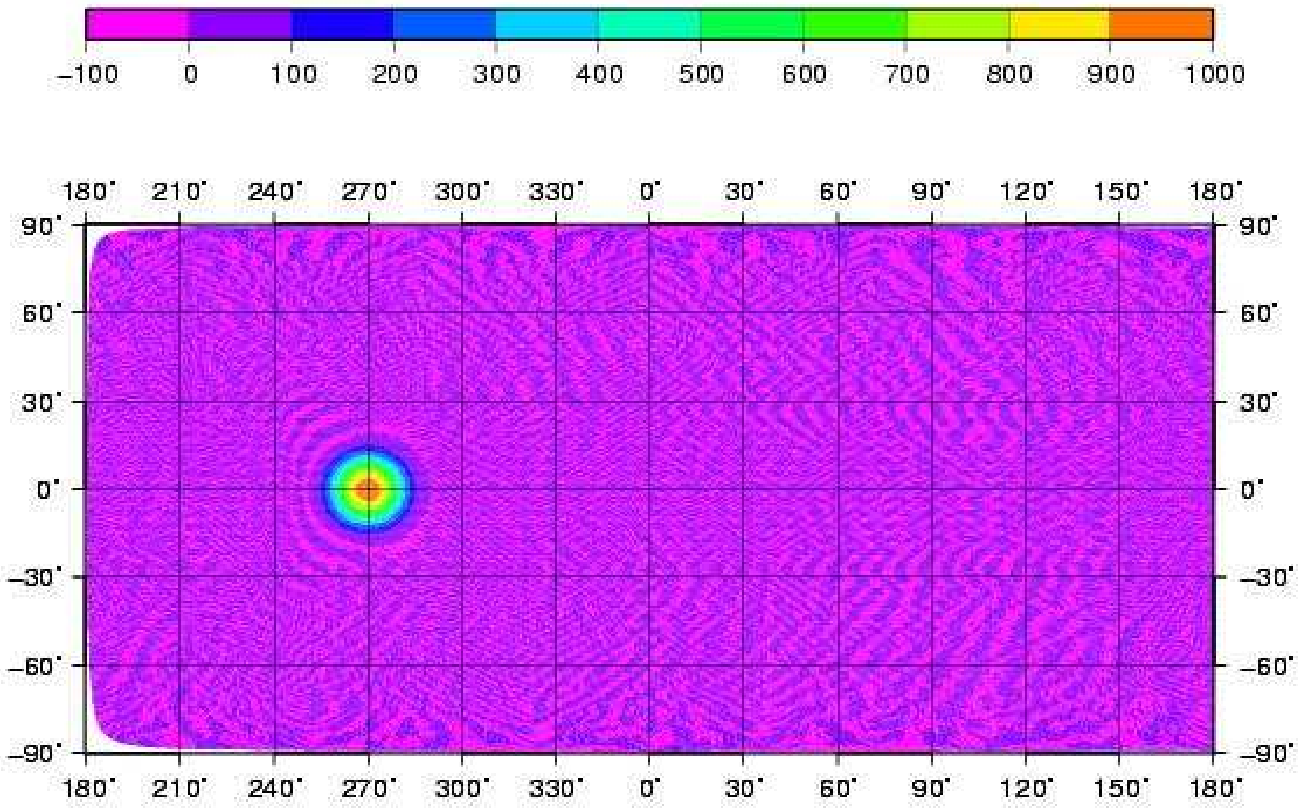
No Diffusion, Forecasted Divergence After 2,5h



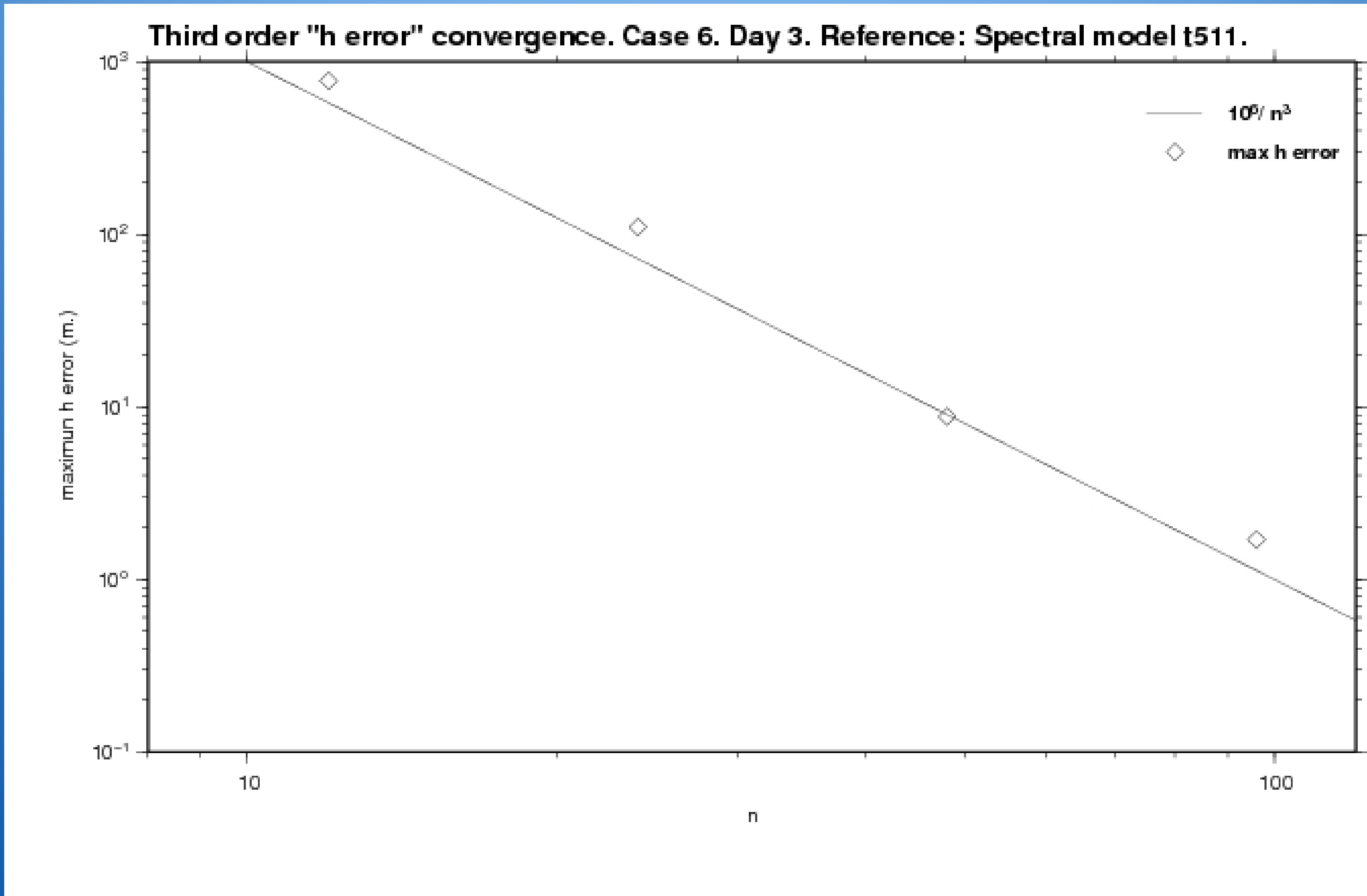
Solid Body Rotation



Homogeneous Adavactionadvection



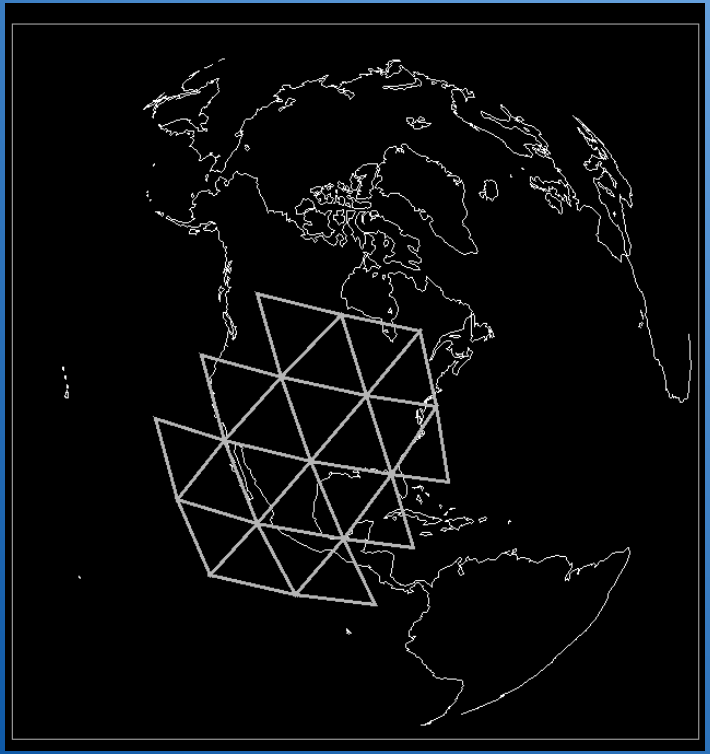
Third Order Convergence of Shallow Water Model at Day 3

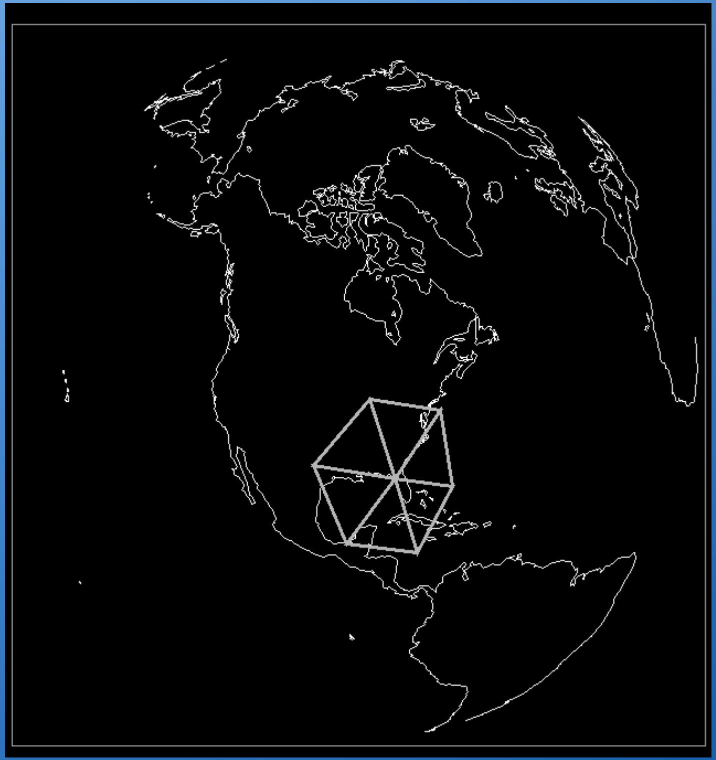


Grid Stencils Baumgardner

- Edges grid Edges grid
- Order3 Order 2

Redundancy 19:10 Redundancy 7:6 or 6:6





High and low order mass formula

Spectral, spectral elements, finite elements:

$$mass\_element = \int_{dF} spectral\_rep\_of\_density \circ dx$$

Finite Difference:

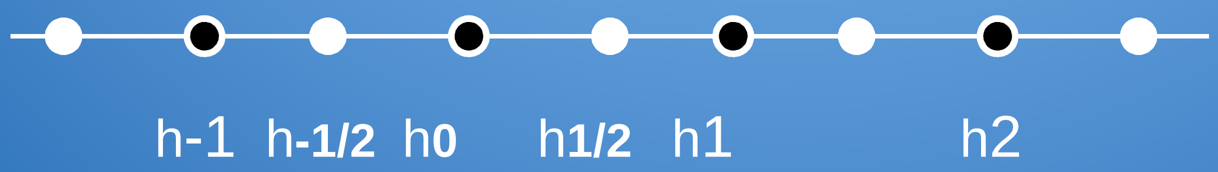
$$mass\_element = \int_{dF} \rho dx \approx \rho_i dx$$

Question: is FD conservation possible with a high order approximation of the density equation?

Definition of flux based conservation schemes

$$\frac{\partial h}{\partial t} = \frac{\partial uh}{\partial x}; \; f = uh \; ; u = 1$$

$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial x}; \; f = h$$



Finite difference flux based conservation:

$$dx\frac{\partial h_0}{\partial t} = h_{1/2} - h_{-1/2}; \Rightarrow \frac{\partial H}{\partial t} = \sum_i dx\frac{\partial h_i}{\partial t} = 0$$

How to define point fluxes $f^*_{-1/2}, f^*_{1/2}$

Stencil: $i \in \{-2,-1,0,1,2\}$

Point flux definition: $f^*_{-1/2} = \sum_i a_i f_i;\ f^*_{1/2} = \sum_i b_i f_i$

Equation for the ai ,bi: $(f^*_{1/2} - f^*_{-1/2})/dx - \dfrac{\partial f}{\partial x} = o(dx^5)$

for all polynomial functions $f(x) = x^n, n \in \{0,1,2,3,.....\}$

Simple example in 1-d

(Only for **regular** grids or the high resolution limit)

$$fp_{i+1} = f_{i+1}\frac{4}{3} - (f_{i+2} + f_i)\frac{1}{6};$$

$$fp_{i-1} = f_{i-1}\frac{4}{3} - (f_{i-2} + f_i)\frac{1}{6}$$

$$\frac{\partial f}{\partial x} 2\,dx = fp_{i+1} - fp_{i-1} = \frac{4}{3}(f_{i+1} - f_{i-1})$$

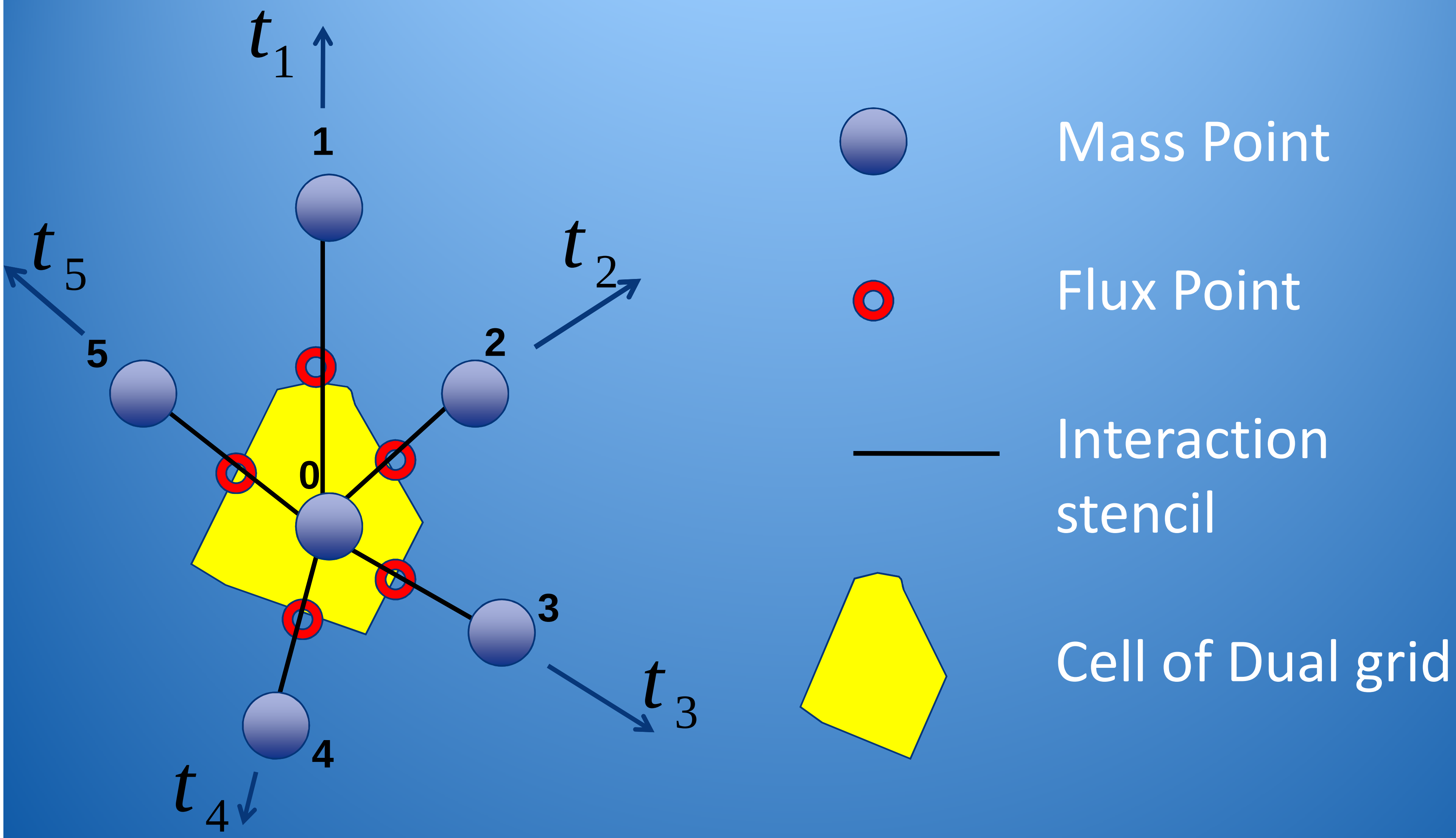
$$-\frac{1}{3}(f_{i+2} - f_{i-2})/2 + o(x^5)$$

Properties of flux based conservation schemes

- If the h**1/2** are exactly computed by spectral method, the flux based scheme is of order 2 at most
- If the h**1/2** are computed by fourth order interpolation from the hi, the flux based scheme is of order 2 at most
- **If h1/2 deviate from the exact value by more than order 2, it is possible to define them in such a way to result into a 3rd or 4th order approximation of the density equation**. Such coarsely approximated fluxes are called **point fluxes**.

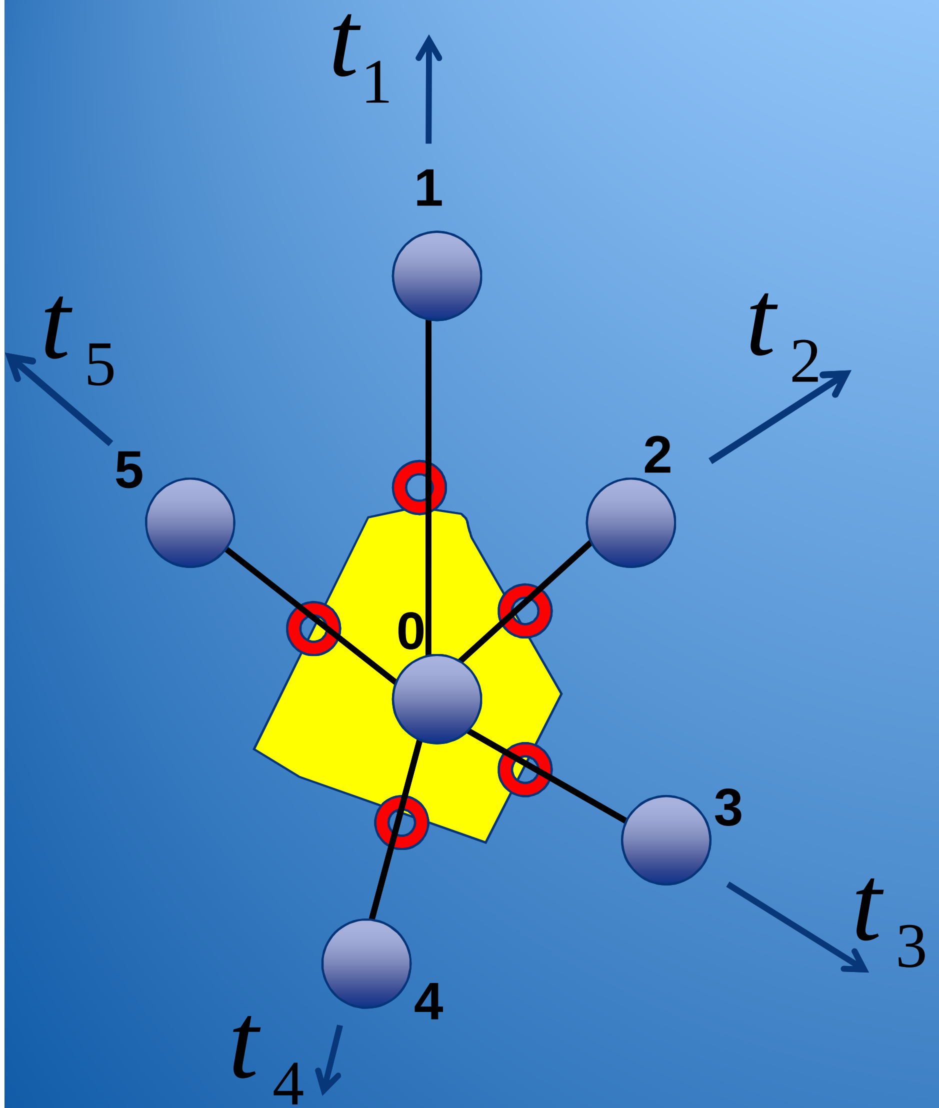
- The example generalises to the **<u>regular</u>** square and hexagon
- To derive similar point fluxes for irregular grids is not trivial
- The formula for the irregular grid to be given are **not** necessarily identical to that above when specialised to the regular case
- **The grid regularisation procedures (spring dynamic) are not necessary when using the Point fluxes for third or second order for the irregular grid**

Irregular Pentagonal Cell



The grid is called quasi regular, if stencil lines cut the sides of the dual cell in half

Irregular Pentagonal Cell



t_1 : directional parameter for line 0,1 etc

f_1 : point flux for point on line 0,1 etc

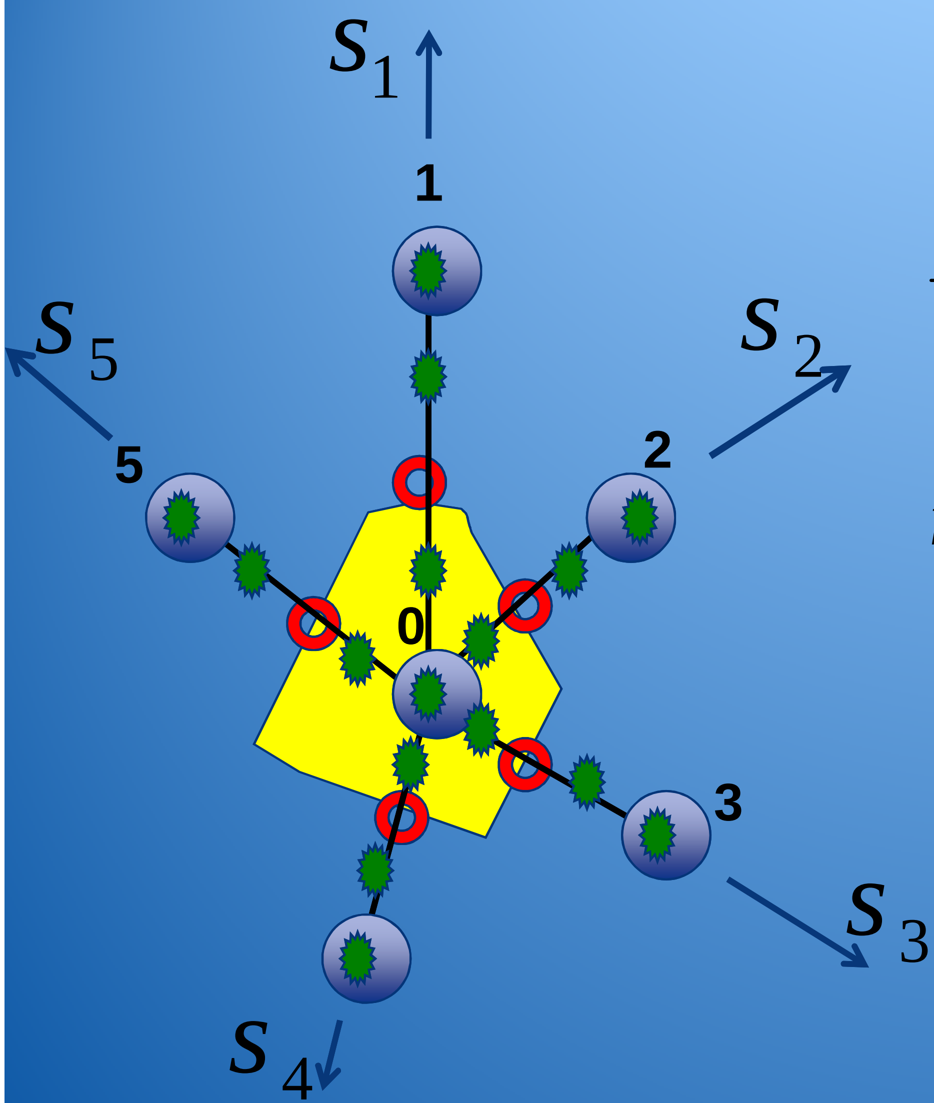
ρ_1 : density derivative for point 0 etc

w_1 : weight to be defined for the point flux

$\dfrac{\delta f_i}{\delta t_i}$: directional derivative

The definition of weights and point fluxes will be obtained by the geometric and collocation formula . This is non trivial even in 1-d for irregular resolution.

Irregular Pentagonal Cell



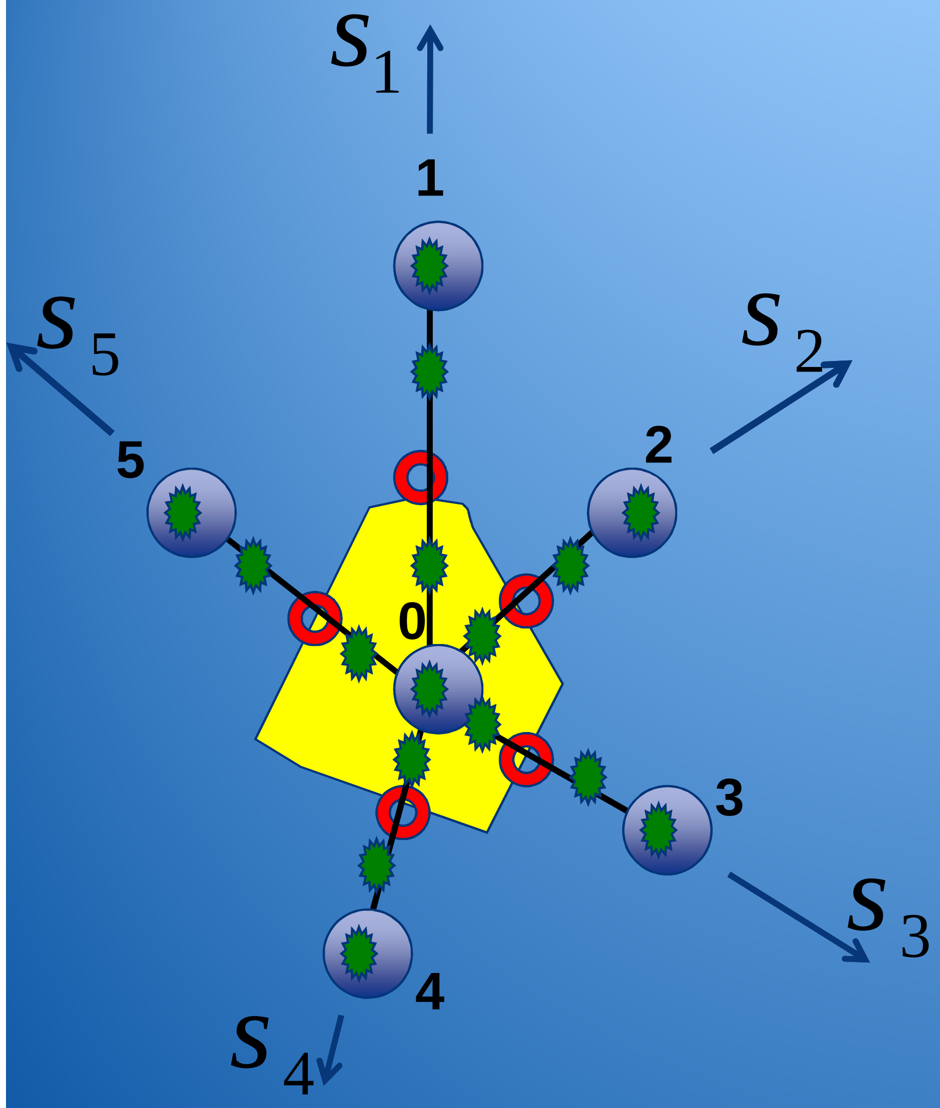
Geometric Formula, derived from Stokes Eq:

$$\frac{\partial \rho}{\partial t} = \sum_i w_i = \sum w_i \overleftrightarrow{f}_i \quad \frac{\delta f}{\delta s_i}(P_0) + w_i^{'} iht_i$$

$$iht = \frac{\delta f}{\delta s_i^{\perp}}(P_0)$$

The inhomogeneous term *iht* is different from 0, when the grid is not quasi regular

Irregular Pentagonal Cell

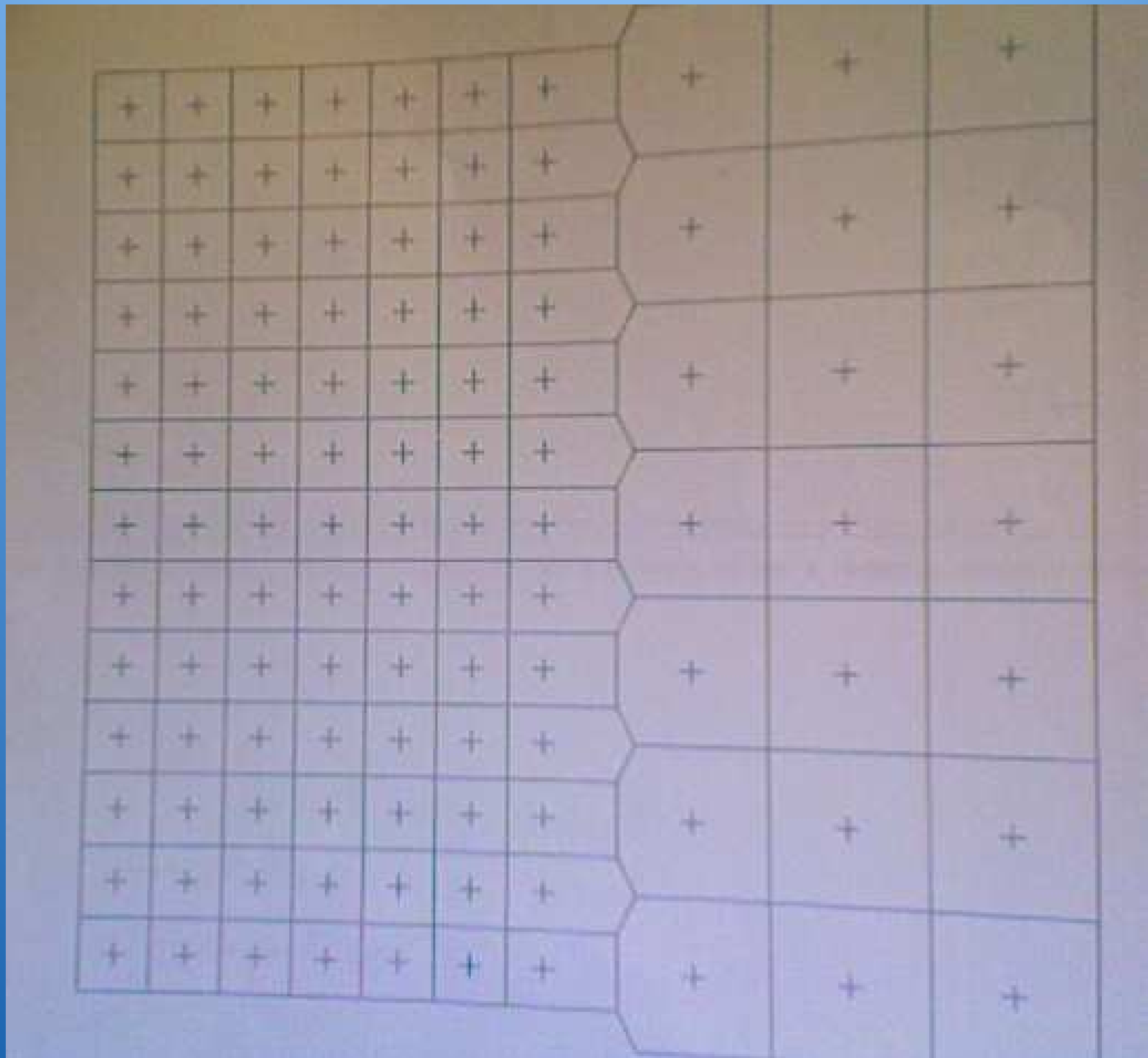


The scheme defined above is conserving, if the point fluxes defined above for points 0 and 1 etc compensate. This is achieved by defining the derivatives by collocation points obtained by interpolation or Galerkin methods.

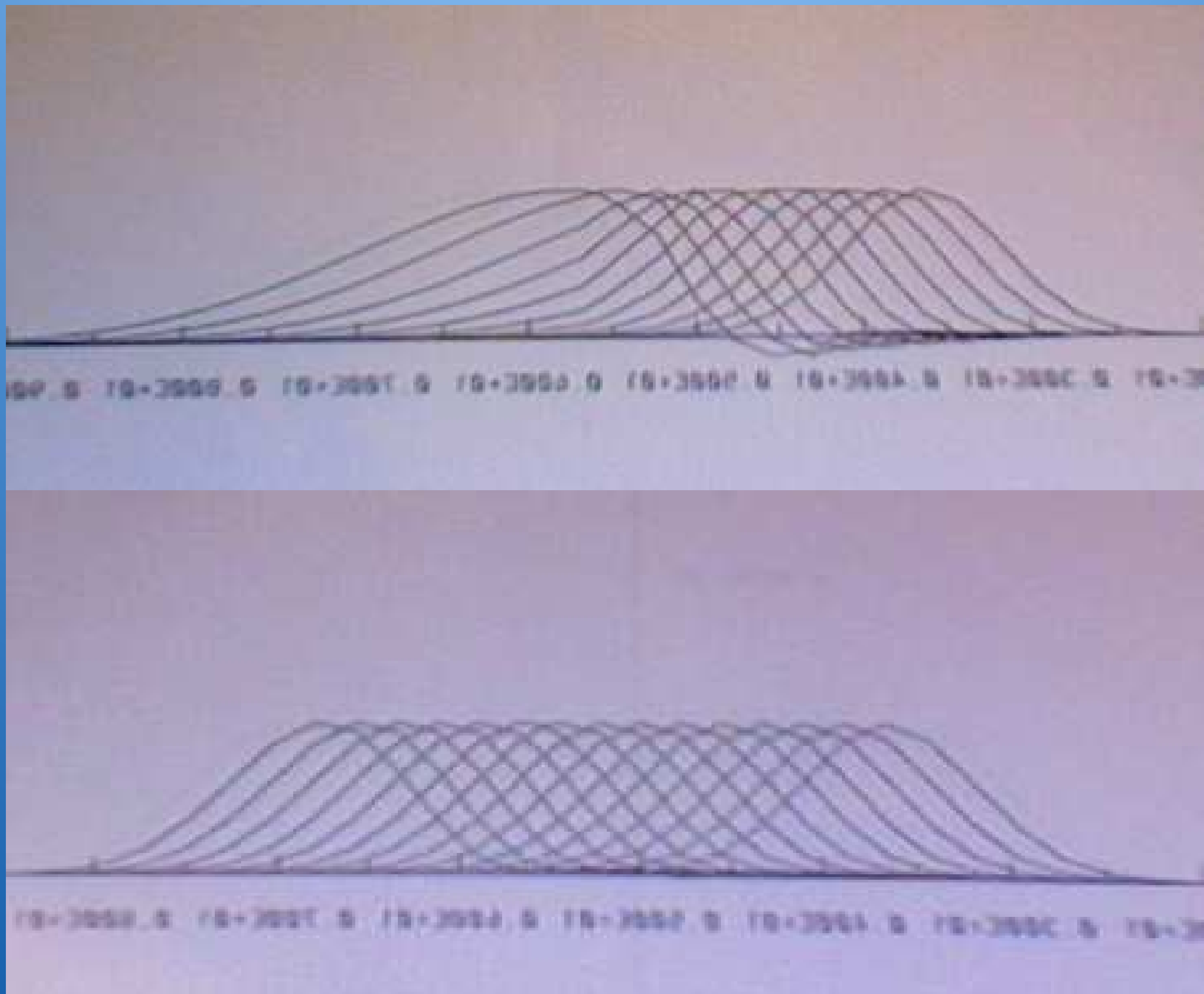
1. In an irregular grid even order 2 in 1-d is non trivial
2. It is a generalisation of Arakawa method, but all interpolations and differentiations must be done in third (second) Order.

Dual Cells with Pentagons

2-d Example: RESOLUTION CHANGE USING



1-d Example
3-rd order scheme with conservation



Conclusions

- In an irregular cell structure point fluxes can be defined as linear functions of surrounding mass or flux points, leading to conserving second or third order schemes.
- Each point flux uses only a small part of the (large) Baumgardner stencil.
- Point fluxes cannot be accurate approximations of fluxes, such as high order interpolations of grid values of fluxes.
- The schemes are free of grid generated noise (Bulls eyes on irregular grids.

Conclusions (Cnt)

- The definition of point fluxes is not unique, therefore other restraints, such as desired behaviour of div 0 waves, can be investigated
- The schemes can be implemented easily, when coordinate of points and the neighbourhood table for mass and flux points as well as the neighbourhood information (Stencil) is available.
- Cut cells can be brought from first to higher order (for the cells near boundaries).
- Problems encountered with irregular and adaptive resolution can often be traced back to discretisation crimes, even hanging nodes.