The Application of the Discontinuous Galerkin Method to a 2d Mesoscale Atmospheric Model

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DWD-Workshop on Non-Hydrostatic Modelling
Bad Orb, Oct 26th 2009

Outline

1. Atmospheric Flow Equations
2. Discontinuous Galerkin Method
3. Mesoscale Atmospheric Model

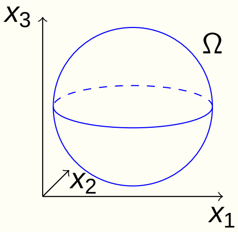
Conservation Laws

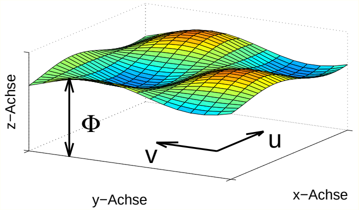
$$
\begin{aligned}
\partial_t q + \mathrm{div}_\Omega f(q,d) &= r && \text{in} \quad \Omega \times \mathbb{R}^+, \\
C(q,d) &= 0 && \text{in} \quad \Omega \times \mathbb{R}^+, \\
q(x,0) &= q_0 && \text{in} \quad \Omega
\end{aligned}
$$

- Variables: conserved $q(x,t) \in \mathbb{R}^s$, diagnostic $d(x,t) \in \mathbb{R}$
- Flux $f \in \mathbb{R}^{s \times N}$, force $r \in \mathbb{R}^s$
- Constraint $C \in \mathbb{R}$
- Boundary conditions

Spherical Shallow Water Equations

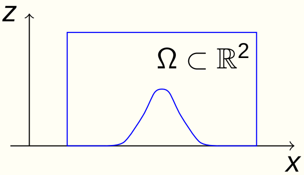
- 2-dimensional surface $\Omega \subset \mathbb{R}^3$
- Conserved variables $q = (\Phi, \Phi u)$
- Flux $f(q) = \begin{pmatrix} \Phi u \\ \Phi u \otimes u + \frac{\Phi^2}{2}\, \mathrm{Id}_3 \end{pmatrix}$





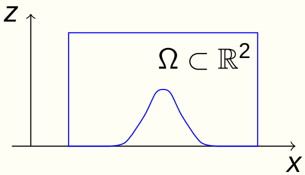
Mesoscale Model – Euler Equations

- $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$
- Conserved variables $q = (\rho, \rho u, \rho w, \rho\theta)$
- Conservation law $\partial_t q + \begin{pmatrix} \rho u & \rho w \\ \rho u^2 + p & \rho u w \\ \rho u w & \rho w^2 + p \\ \rho\theta u & \rho\theta w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -g\rho \\ 0 \end{pmatrix}$



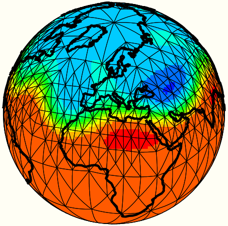
Mesoscale Model – Hydrostatic Equations

- Conserved variables $q = (\rho, \rho u, \rho\theta)$
- Diagnostic variable w
- Conservation law $\partial_t q + \operatorname{div} \begin{pmatrix} \rho u & \rho w \\ \rho u^2 + p & \rho u w \\ \rho\theta u & \rho\theta w \end{pmatrix} = 0$
- Hydrostatic constraint $\partial_z p = -g\rho$

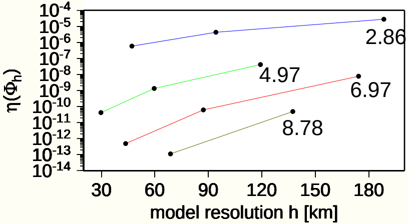


DG for Atmospheric Applications

- Spherical SWE: Giraldo ’02, Nair ’05, Giraldo ’06
- Mesoscale Euler codes: Giraldo ’08, Restelli ’09



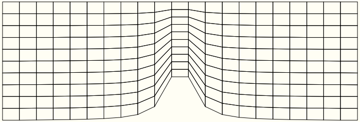
SWE convergence

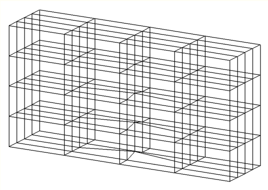


L. ’08

Discontinuous Function Space

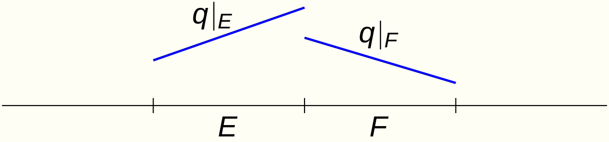
- Computational Grid, elements $E \subset \Omega$





- Function space

$$V = \{\varphi : \Omega \to \mathbb{R} \quad | \quad \varphi|_E \in P^k(E), \quad \forall E\}$$



Space Discretization

$$\partial_t q + \operatorname{div} f(q) = 0$$

- Integral form, $q(t) \in V^s$

$$\int_E \varphi \cdot \partial_t q - f(q) : \nabla_\Omega \varphi \, dx + \int_{\partial E} \varphi \cdot f(q) \cdot \nu_E \, d\sigma \qquad = 0, \quad \forall \varphi \in V^s$$

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- Rusanov (local Lax-Friedrichs) numerical flux

$$\hat{f}(x, q^{in}, q^{out}) = \frac{1}{2} \left[(f(q^{in}) + f(q^{out})) \cdot \nu_E - \lambda (q^{out} - q^{in}) \right]$$

Spatial Accuracy

Semi-discrete solution

$$(\partial_t q_h, \varphi)_{L^2(\Omega)} = [f]_{DG}(q_h, \varphi), \quad \forall \varphi \in V^s$$

- Linear f, dimension $n = 1$, scalar $s = 1$,
- Tensor product polynomials P^k

$$\|q_h(T) - q(T)\|_{L^2(\Omega)} = \mathcal{O}(\Delta x^{k+1})$$

LeSaint '74

Nodal Polynomial Basis

- Polynomials $P^k(E)$, $N_k = \dim P^k$
- Lagrange basis $(\varphi_i)_{i=1,..,N_k}$, collocation points $x_1, .., x_{N_k}$
- Data points $\bar{d} = (d_1, .., d_{N_k})$

$$\mathcal{L}_{x_i} : \mathbb{R}^{N_k} \to P^k, \quad \mathcal{L}_{x_i}\bar{d} = \sum_{i=1}^{N_k} d_i \varphi_i(x)$$

$$\|\mathcal{L}_{x_i}\bar{d}\|_\infty \leq \|\mathcal{L}_{x_i}\| \|\bar{d}\|_\infty$$

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$$\|\mathcal{L}_{x_i}\bar{d}\|_\infty \le \|\mathcal{L}_{x_i}\| \|\bar{d}\|_\infty$$

- Small Lebesgue constant $\|\mathcal{L}_{x_i}\|$
  - $(0,1)^n$: Gauss-Lobatto points
  - Triangle: (elliptic) Fekete points

Taylor '00, Hesthaven '98

Time Discretization

$$q(x,t) = \sum_{i=1}^{\dim V^s} q_i(t)\varphi_i(x)$$

$$(\partial_t q, \varphi)_{L^2} = [f]_{DG}(q,\varphi), \quad \forall \varphi \in V^s,$$

$$\sum_i \partial_t q_i (\varphi_i, \varphi_j)_{L^2} = [f]_{DG}(q,\varphi_j), \quad j = 1,..,\dim V^s$$

- RKM: Explicit strong stability preserving Runge-Kutta method
  - non-linear f, 1d, scalar, slope limiter: convergence (Cockburn/Shu)
- BDF2/3: IMEX backward difference formula

$$(\partial_t q, \varphi)_{L^2} = [f - l]_{DG}(q,\varphi) + [l]_{DG}(q,\varphi)$$

1. Atmospheric Flow Equations
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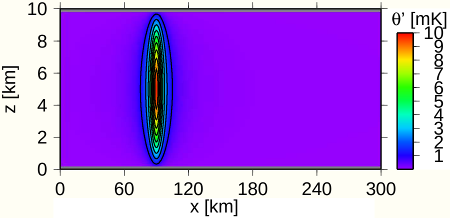
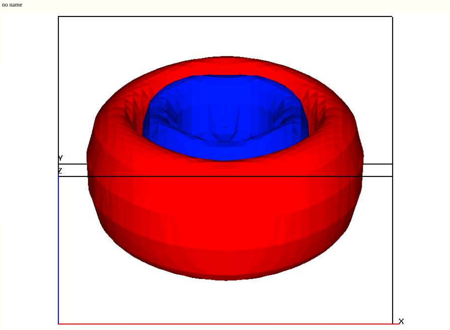
Gravity Wave Setup

- Domain $\Omega = (0, 300\text{km}) \times (0, 300\text{km}) \times (0, 10\text{km})$
- Velocity $u = 20\text{ms}^{-1}$
- Time $T = 3000\text{s}$





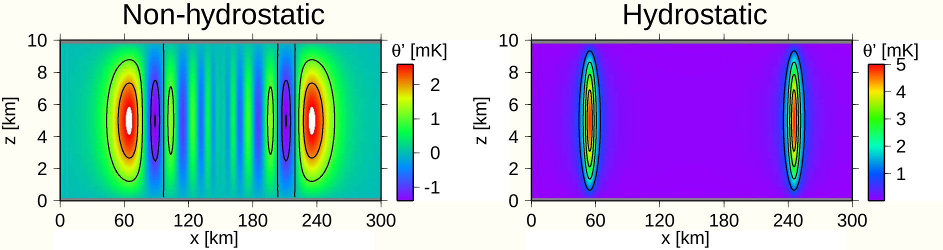
Skamarock '94

Linear Analytic Solution

- Phase velocity

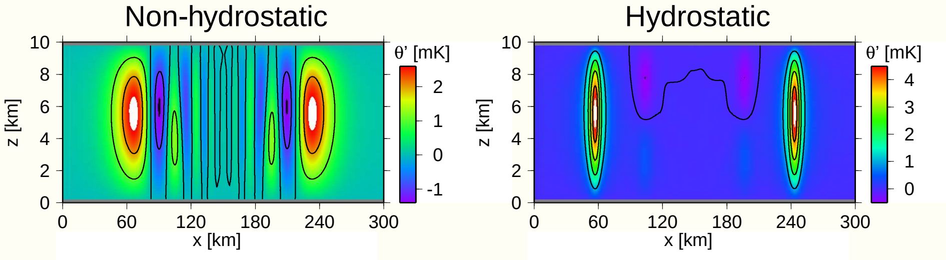
$$c(k) = \frac{N}{\sqrt{\delta k^2 + l^2}}$$

- Hydrostatic switch δ
- Brunt-Väisälä frequency N, wave numbers k, l

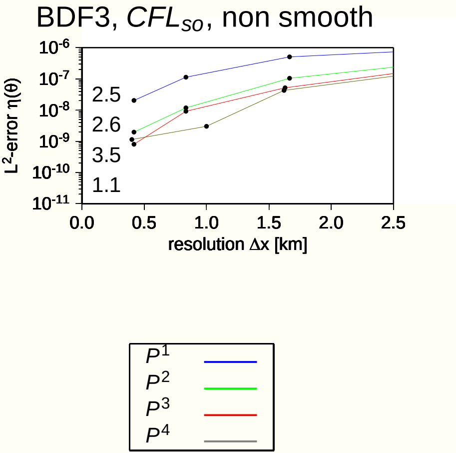


2d Model Results

- $P^k, k = 1, \dots, 4$
- $\Delta x = 200\mathrm{m} \times 200\mathrm{m}$

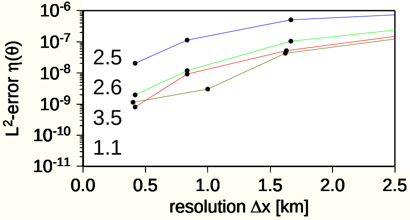


Gravity Wave Convergence

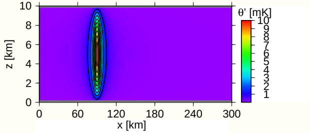


Gravity Wave Convergence

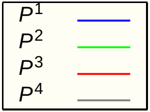
BDF3, CFL_{so}, non smooth



Agnesi mountain profile



$$\theta' = \theta_z(z)\frac{a^2}{a^2 + (x - x_c)^2}$$



Gravity Wave Convergence

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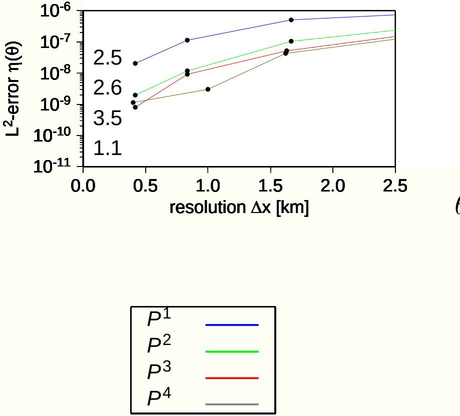

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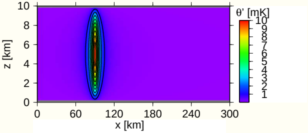
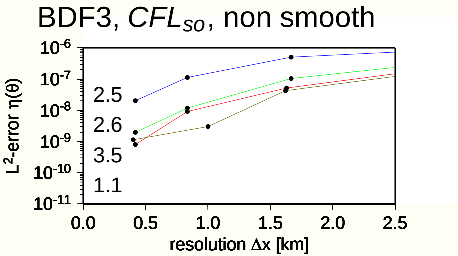
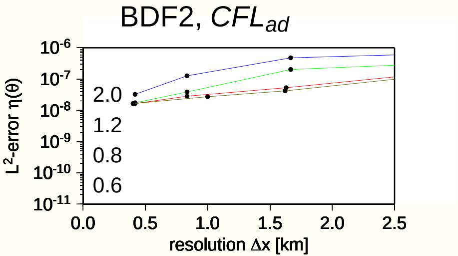
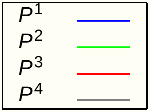


$$\theta' = \theta_z(z)\frac{a^2}{a^2 + (x - x_c)^2}\varphi_c(x - x_c)$$

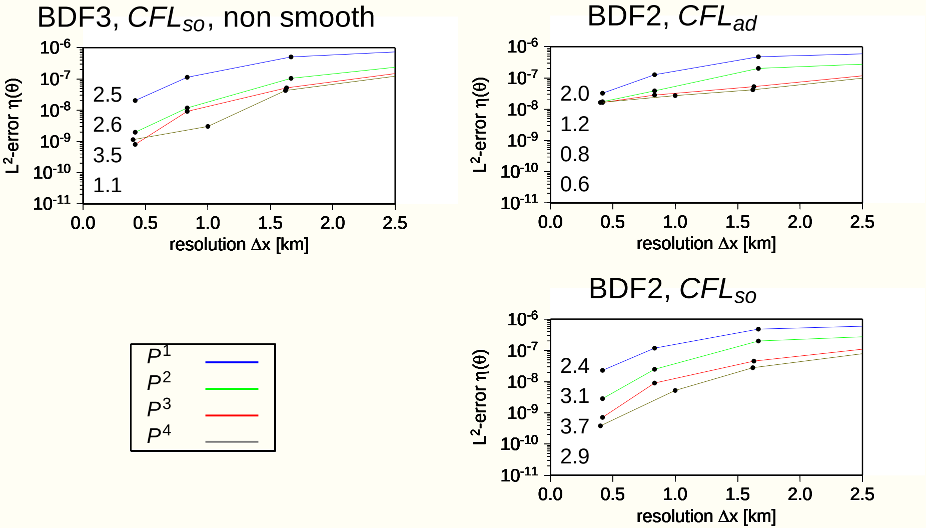
Gravity Wave Convergence





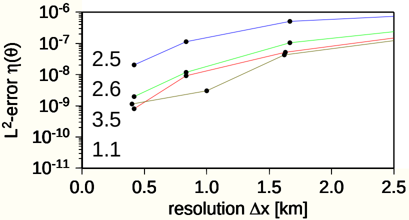


Gravity Wave Convergence

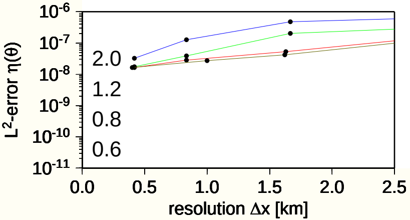


Gravity Wave Convergence

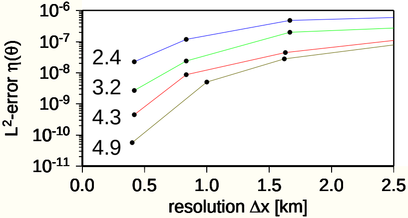
BDF3, CFL_{so}, non smooth



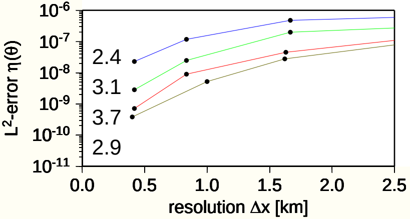
BDF2, CFL_{ad}



BDF3, CFL_{so}



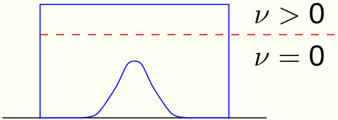
BDF2, CFL_{so}



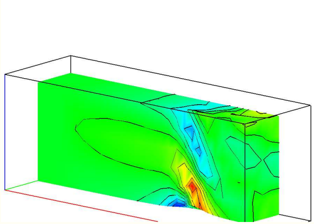
Hydrostatic Mountain Waves

- Domain
  $\Omega = (0, 240\text{km}) \times (0, 80\text{km}) \times (0, 22\text{km})$
- Mountain extent $20\text{km} \times 1\text{m}$
- Lateral inflow-, outflow- conditions
- Non-reflecting boundary

$$\partial_t q + \text{div}\, f(q) = r(q) + \nu(q_{ref} - q)$$



u-perturbation



Model Results

- $\Delta x = 1200\mathrm{m} \times 280\mathrm{m}$
- Steady state waves

Non-hydrostatic | Hydrostatic

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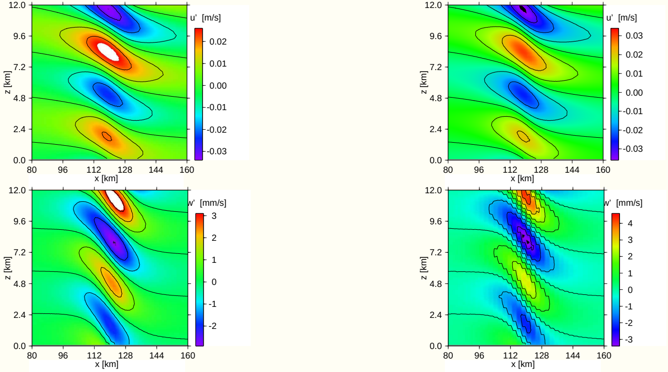

Schär Mountain Waves

- $\Omega \subset (-25, 144\mathrm{km}) \times (0, 30\mathrm{km})$
- 5 mountains: $h_x = 4\mathrm{km}$, $h_z = 240\mathrm{m}$

Non-hydrostatic | Hydrostatic

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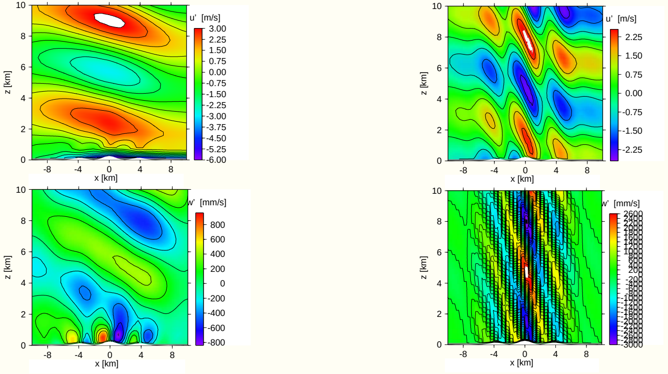
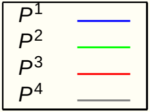
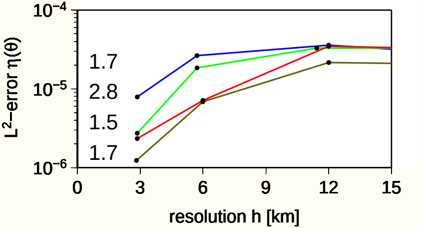


Mountain Wave Convergence

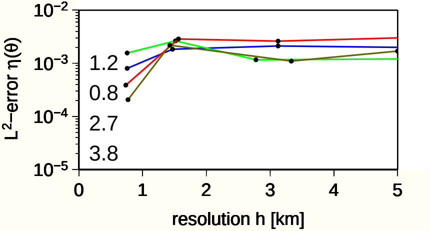
- Non-hydrostatic system
- Steady state solution
- Resolve scales ↔ convergence



Hydrostatic mountain



Schär mountain



- Conservation laws for atmospheric flow
  - Shallow water equations
  - 2d-Euler equations
  - Hydrostatic equations
- DG method
  - Polynomial order $k = 1, ..., 4$
  - Time integrators: RKM, BDF
- High order convergence results
- Wave dispersion for (non-)hydrostatic systems
- Limits for the hydrostatic system

Workshop Announcement

Solution of Partial Differential Equations on the Sphere 2010

- August 24 - 27, 2010
- Numerical Methods and Applications for Geophysical Flow
- Potsdam, Germany

Alfred Wegener Institute

Sanssouci Castle