Godunov-type transport scheme in ICON

Implementation and results of idealised test cases

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Outline

I. The ICON-grid

II. Transport on unstructured grids

- Introduction into combined semi-Lagrangian finite-Volume thinking

III. Currently implemented transport scheme

- Reconstruction
- Flux integration

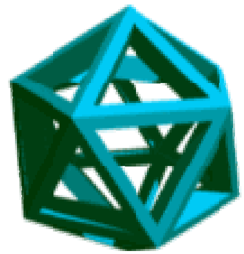
IV. Results of idealised test cases on the sphere

- Solid body rotation
- Deformational flow (static vortex)

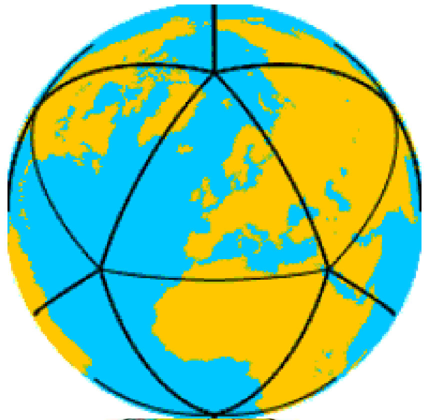
V. Summary and outlook

I. The ICON-grid

Grid topology and geometry

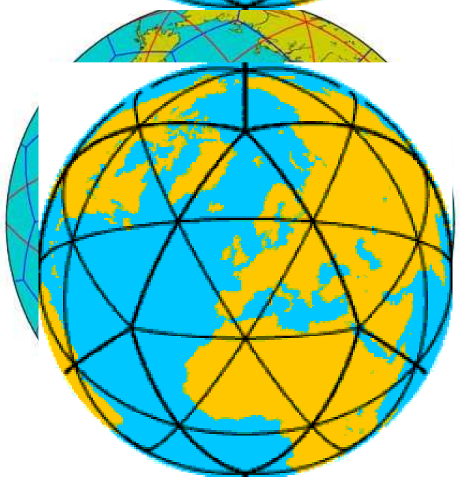


- Inscribe icosahedron inside the unit sphere
- The 12 vertices touching the surface define the basic mesh consisting of 20 spherical triangles.
- Further mesh refinement by one „root division“ followed by successive bisections (connect midpoints of the edges for each triangle by great circle arcs)

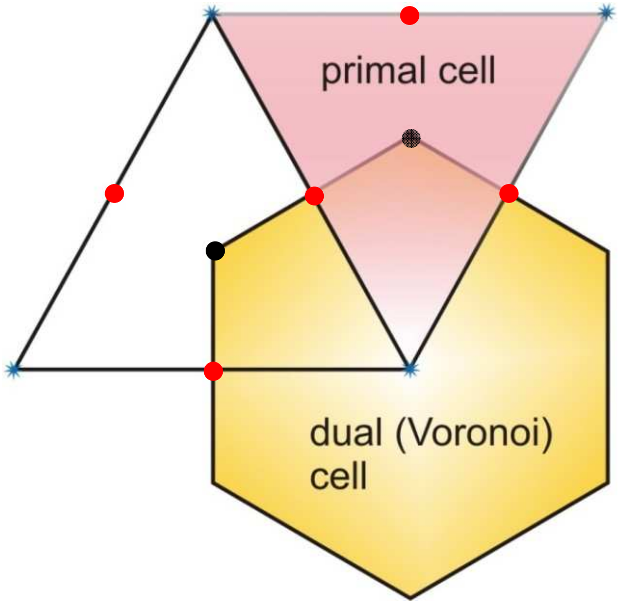


- Primal (Delaunay) grid: triangles
- Dual (Voronoi) grid: hexagons

  (+ 12 pentagons at the icosahedron vertices)



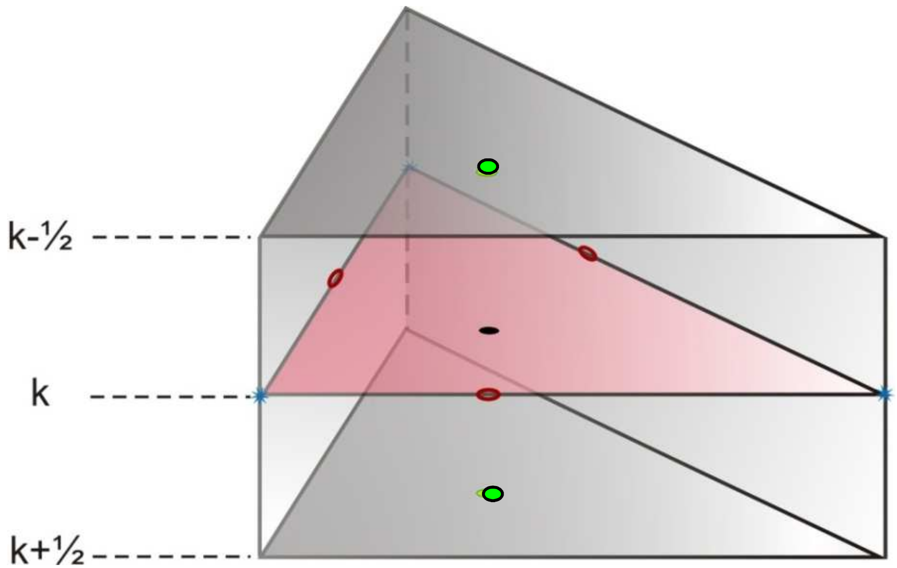
3D arrangement of the discrete variables



horizontal

C-type staggering

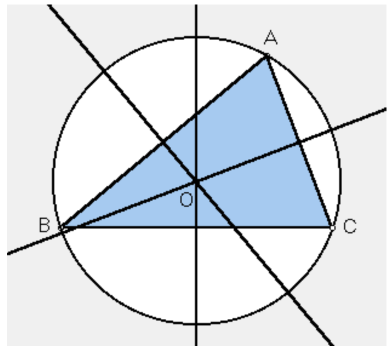
- ● T, q, p, Φ
- ● v_n
- ✳ $\vec{k} \bullet (\vec{\nabla} \times \vec{v})$
- ● $\dot{\eta}\frac{\partial p}{\partial \eta}, \Phi, p$



vertical

- Cell center: center of triangle circumcircle

  $\Rightarrow$ Arc connecting two mass points is orthogonal to and bisects triangle edge



II. Introduction into combined semi-Lagrangian finite-volume thinking

Problem formulation (horizontal)

- Find solution to

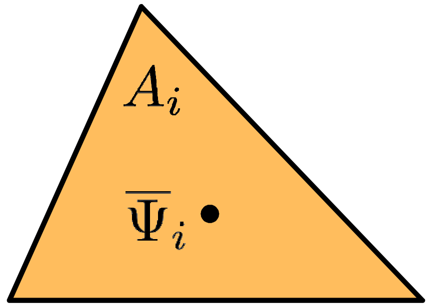
$$\frac{\partial \Psi}{\partial t} + \nabla \cdot (\vec{v}\,\Psi) = 0$$

$$\begin{aligned} \Psi &= \rho\, q & &\text{, if non-hydrostatic} \\ \Psi &= \Delta p\, q & &\text{, if hydrostatic} \end{aligned}$$

on the sphere, using finite-Volume technique.

- Define control volume
  - Discrete value at mass point is defined to be the average of the subgrid dist. over the control volume

$$\overline{\Psi}_i^{\,n} = \frac{1}{A_i} \iint_{A_i} \Psi(x, y, t_0)\, \mathrm{d}A$$



A_i : control volume area

- Given $\overline{\Psi}_i$ at time t_0 we seek for a new set of $\overline{\Psi}_i$ at time $t_1 = t_0 + \Delta t$ as an approximate solution after a short time of transport.

Problem formulation

- Formally, the solution can be derived by integrating the continuity equation over the time interval $[t_0,t_1]$ and the control volume A_i

$$\int_{t_0}^{t_1} \iint_{A_i} \frac{\partial \Psi}{\partial t}\, \mathrm{d}A\, \mathrm{d}t = -\int_{t_0}^{t_1} \iint_{A_i} \vec{\nabla} \cdot (\vec{v}\, \Psi)\, \mathrm{d}A\, \mathrm{d}t$$

using Gauss-theorem

$$\overline{\Psi}_i^{n+1} - \overline{\Psi}_i^{n} = -\frac{1}{A_i} \int_{t_0}^{t_1} \oint_{\partial A_i} (\Psi \vec{v} \cdot \vec{n}_i)\, \mathrm{d}l\, \mathrm{d}t$$

Assume triangular control volume and $\vec{v} = \mathrm{const} \quad \text{for } t\, \varepsilon\, [t_0, t_1]$

$$\overline{\Psi}_i^{n+1} = \overline{\Psi}_i^{n} - \frac{1}{A_i} \sum_{k=1}^{3} \vec{F}_{ik} \cdot \vec{n}_{ik} \qquad \text{, with} \quad \vec{F}_{ik} = \left(\iint_{A_{ik}} \Psi\, \mathrm{d}A \right) \cdot \vec{\hat{n}}_{ik}$$

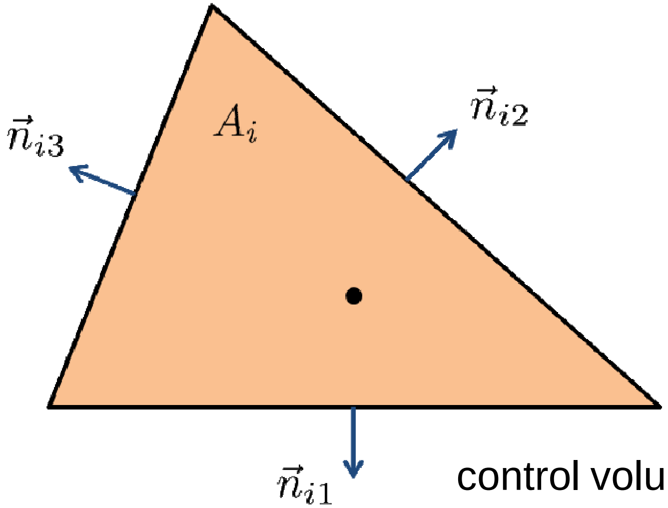
- No approximation as long as we know the subgrid distribution Ψ and A_{ik} analytically; except for assumption of $\vec{v} = \mathrm{const} \quad \text{for } t\, \varepsilon\, [t_0, t_1]$

DWD

What does it mean graphically ?

$$\overline{\Psi}_i^{n+1} = \overline{\Psi}_i^n - \frac{1}{A_i} \sum_{k=1}^{3} \vec{F}_{ik} \cdot \vec{n}_{ik} \qquad \text{, with} \quad \vec{F}_{ik} = \left(\iint_{A_{ik}} \Psi \, \mathrm{d}A \right) \cdot \vec{\hat{n}}_{ik}$$

$\vec{n}_{ik}, \; \vec{\hat{n}}_{ik}$: Unit normal pointing outward of A_i and A_{ik}, respectively

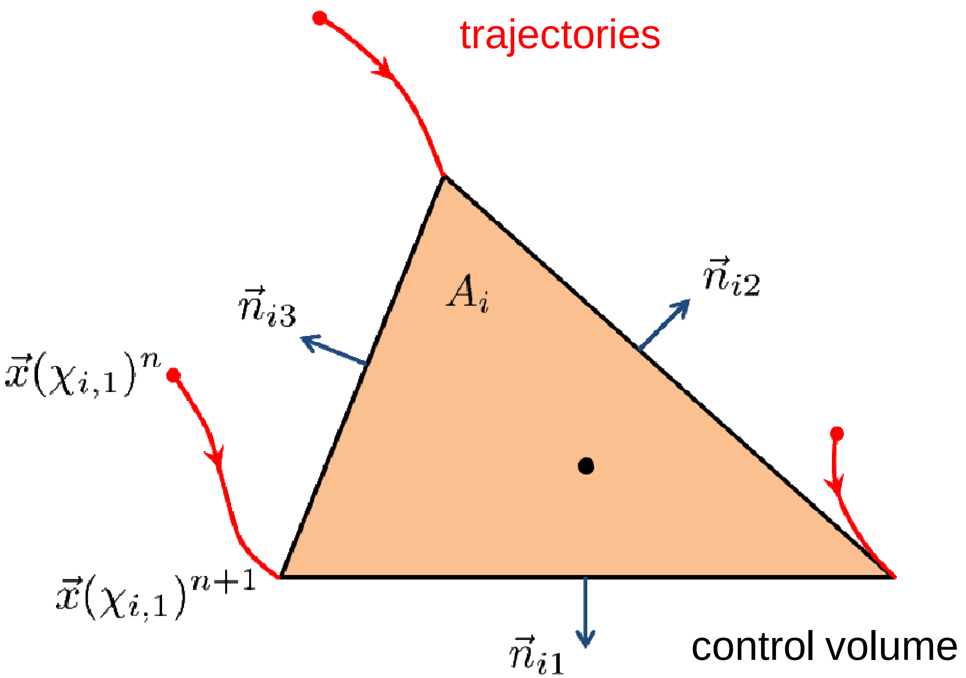


control volume

What does it mean graphically ?

$$\overline{\Psi}_i^{n+1} = \overline{\Psi}_i^n - \frac{1}{A_i}\sum_{k=1}^{3} \vec{F}_{ik}\cdot\vec{n}_{ik} \qquad \text{, with} \quad \vec{F}_{ik} = \left(\iint_{A_{ik}} \Psi \, \mathrm{d}A\right)\cdot\vec{\hat{n}}_{ik}$$

$\vec{n}_{ik},\ \vec{\hat{n}}_{ik}$: Unit normal pointing outward of A_i and A_{ik}, respectively



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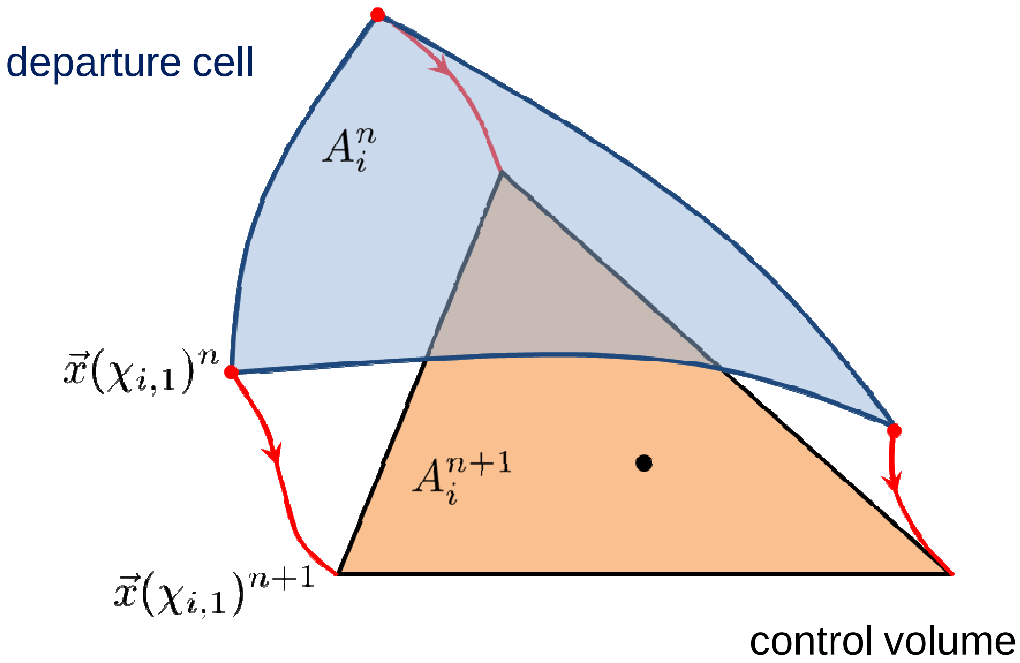
$$\overline{\Psi}_i^{n+1} = \overline{\Psi}_i^n - \frac{1}{A_i}\sum_{k=1}^{3} \vec{F}_{ik}\cdot\vec{n}_{ik} \qquad \text{, with} \quad \vec{F}_{ik} = \left(\iint_{A_{ik}} \Psi\,\mathrm{d}A\right)\cdot\vec{\hat{n}}_{ik}$$

$\vec{n}_{ik},\ \vec{\hat{n}}_{ik}$: Unit normal pointing outward of A_i and A_{ik}, respectively



semi-Lagrangian:

$$A_i^n\,\overline{\Psi}_{A_i^n} = A_i^{n+1}\,\overline{\Psi}_i^{n+1}$$

↓

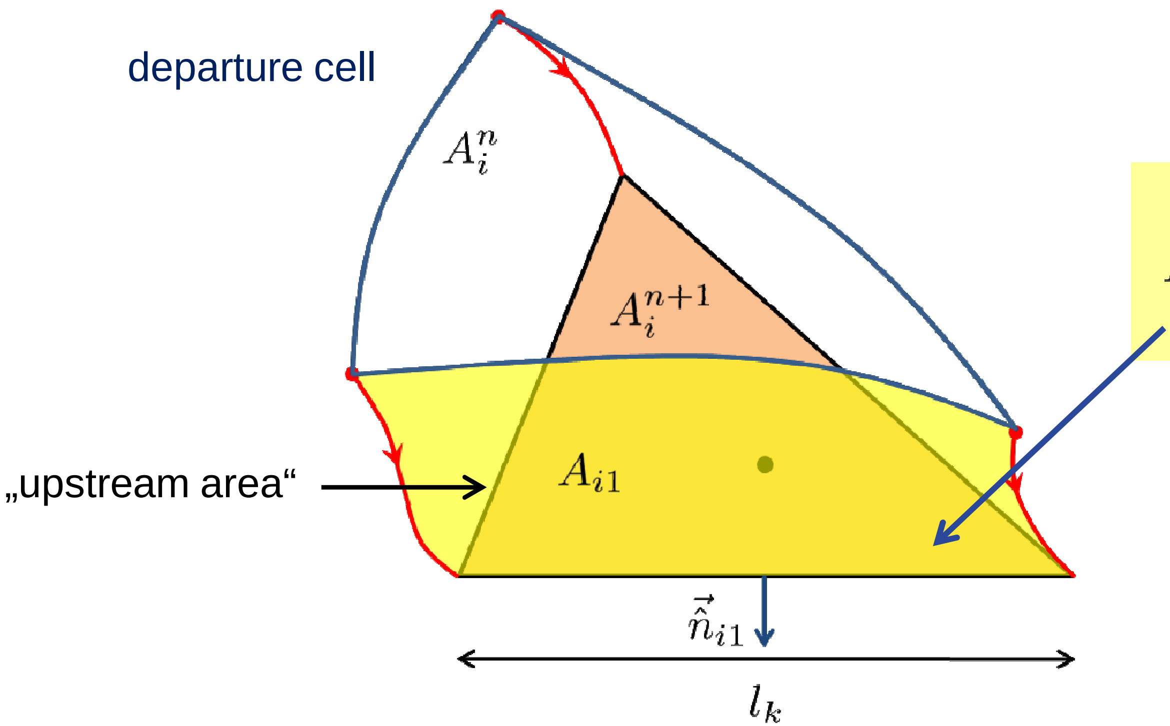
issue is down to calculating

$$\iint_{A_i^n} \Psi(x,y,t_0)\,\mathrm{d}A$$

What does it mean graphically ?

$$\overline{\Psi}_i^{n+1} = \overline{\Psi}_i^n - \frac{1}{A_i} \sum_{k=1}^{3} \vec{F}_{ik} \cdot \vec{n}_{ik} \qquad \text{, with} \quad \vec{F}_{ik} = \left(\iint\limits_{A_{ik}} \Psi \, \mathrm{d}A \right) \cdot \vec{\hat{n}}_{ik}$$

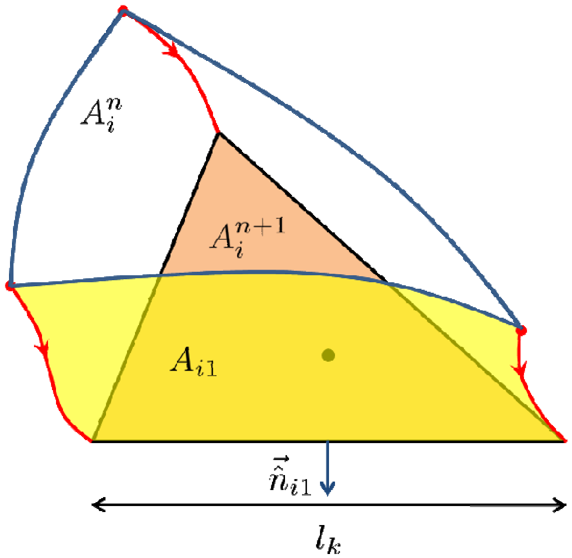
$\vec{n}_{ik}, \ \vec{\hat{n}}_{ik}$: Unit normal pointing outward of A_i and A_{ik}, respectively



$$\vec{F}_{i,1} = \left(\iint\limits_{A_{i1}} \Psi(x, y, t_0) \, \mathrm{d}A \right) \cdot \vec{\hat{n}}_{i1}$$

Algorithm

The numerical algorithm consists of three major steps



1. Determine the upstream area A_{ik} for the k^{th} edge
2. Determine the subgrid distribution $\Psi(x, y, t_0)$ for control volume i with cell average $\overline{\Psi}_i^n$
3. Integrate the subgrid distribution $\Psi(x, y, t_0)$ over the area A_{ik}.

$$\iint_{A_{ik}} \Psi(x, y, t_0)\,\mathrm{d}A$$

plug in

$$\overline{\Psi}_i^{n+1} = \overline{\Psi}_i^n - \frac{1}{A_i}\sum_{k=1}^{3} \vec{F}_{ik}\cdot\vec{n}_{ik} \qquad \text{, with} \quad \vec{F}_{ik} = \left(\iint_{A_{ik}} \Psi\,\mathrm{d}A\right)\cdot\vec{\hat{n}}_{ik}$$

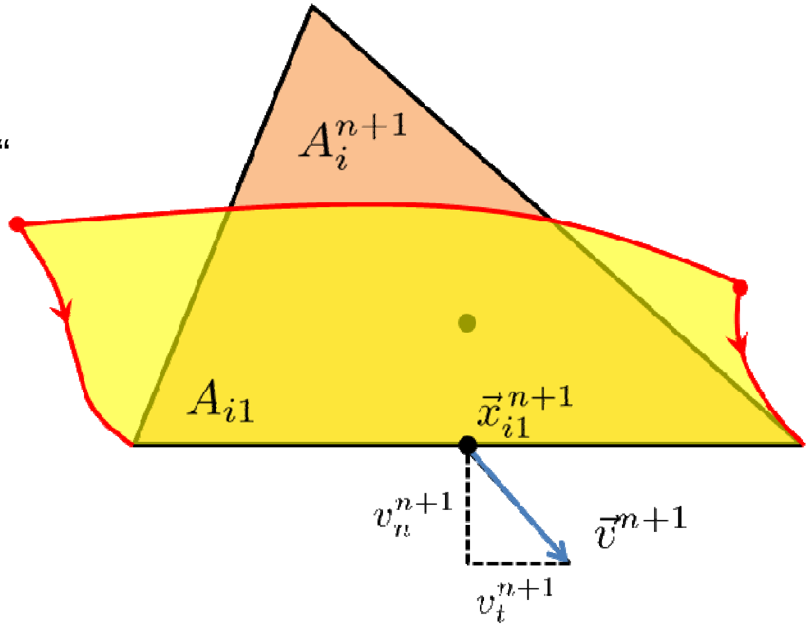
III. Currently implemented transport scheme

1. Approximation of the upstream area A_{ik}

normal component v_n^{n+1} : prognostic variable

tangential component v_t^{n+1} : reconstructed via RBF (radial basis functions)

„upstream area"

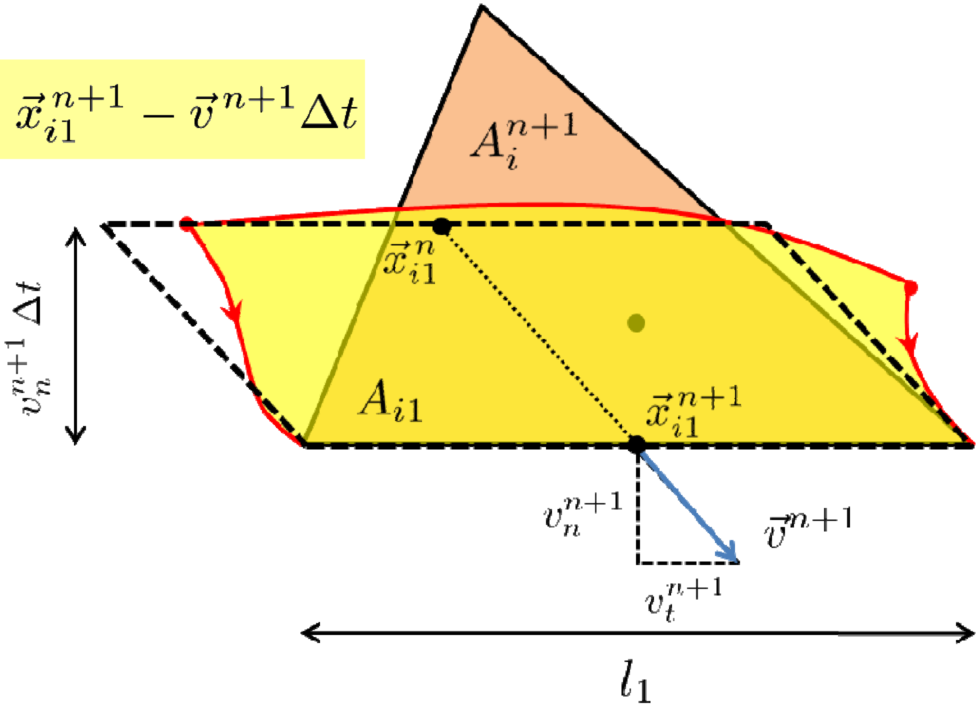


1. Approximation of the upstream area A_{ik}

normal component v_n^{n+1} : prognostic variable

tangential component v_t^{n+1} : reconstructed via RBF (radial basis functions)

$$\vec{x}_{i1}^{\,n} = \vec{x}_{i1}^{\,n+1} - \vec{v}^{\,n+1}\Delta t$$



analytic area

-------- approximation

2. Approximation of subgrid distribution $\Psi(x,y,t_0)$

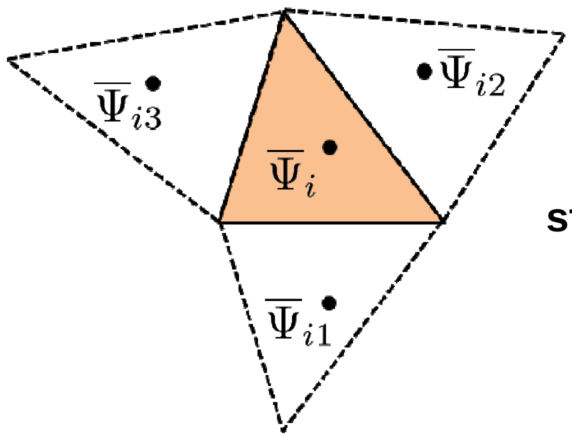
- Piecewise linear approximation for subgrid distribution $\Psi(x, y, t_0)$

$$\Psi^R(\vec{x} - \vec{x}_i) = \Psi|_{\vec{x}_i} + \left.\frac{\partial \Psi}{\partial x}\right|_{\vec{x}_i} (x - x_i) + \left.\frac{\partial \Psi}{\partial y}\right|_{\vec{x}_i} (y - y_i)$$

(x_i, y_i) : mass point of control volume i

unknown

- unknowns are chosen to conserve the mean value $\overline{\Psi}_i$ in the control volume i and to minimize the error in predicting the mean values $\overline{\Psi}_{i1}$, $\overline{\Psi}_{i2}$, $\overline{\Psi}_{i3}$ for control volumes in the stencil.



stencil

- For each control volume i:

4 equations for 3 unknowns $\Psi|_{\vec{x}_i}$, $\left.\frac{\partial \Psi}{\partial x}\right|_{\vec{x}_i}$, $\left.\frac{\partial \Psi}{\partial y}\right|_{\vec{x}_i}$

weighted least squares reconstruction

3. Integrate subgrid distribution

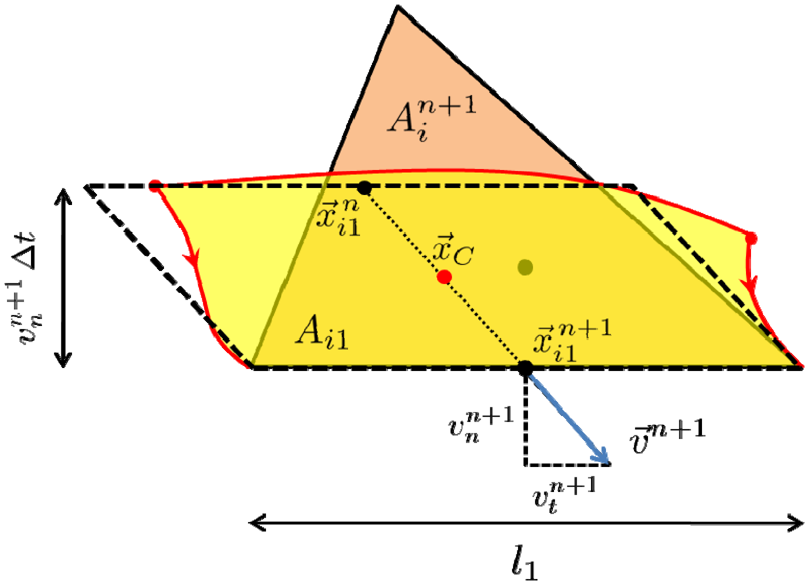
1. Upstream area A_{i1} ✓
2. Subgrid distribution $\Psi(x,y,t_0)$ ✓

3. **Integration**

$$\vec{F}_{i1} \cdot \hat{\vec{n}}_{i1} = \iint\limits_{A_{i1}} \Psi(x,y,t_0)\,\mathrm{d}A = A_{i1}\overline{\Psi}_{A_{i1}}$$

upstream area: $A_{i1} = l_1 \left(v_n^{n+1}\Delta t\right)$

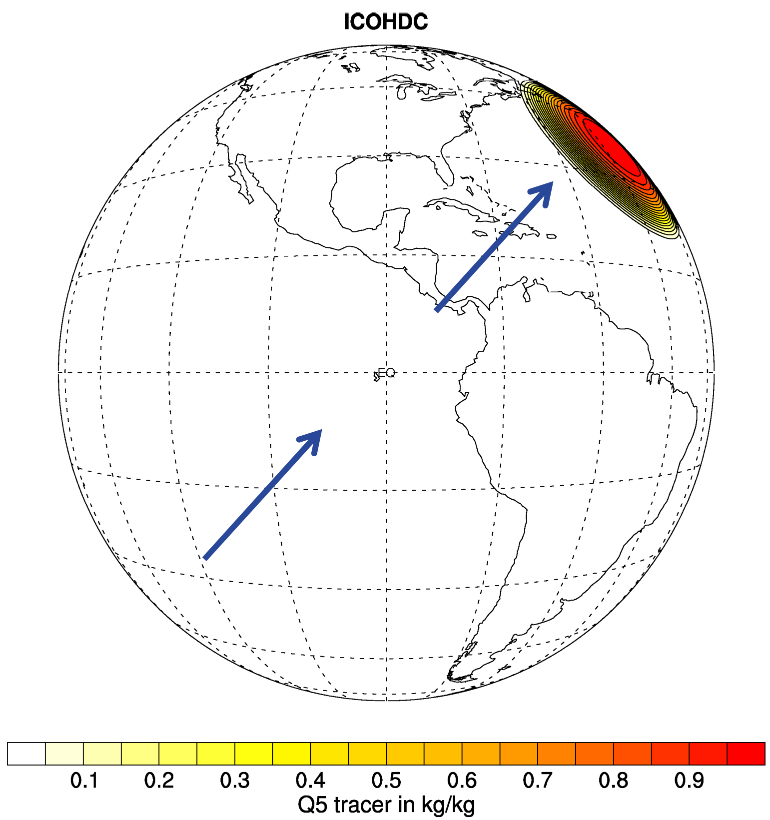
area average:
$$\overline{\Psi}_{A_{i1}} = \Psi^R(\vec{x}_C)$$
$$\vec{x}_C = \vec{x}_{i1}^{\,n+1} - \frac{\Delta t}{2}\vec{v}^{\,n+1}$$ center of mass of parallelogramm



IV. Results of idealised test cases on the sphere

Solid body rotation test case

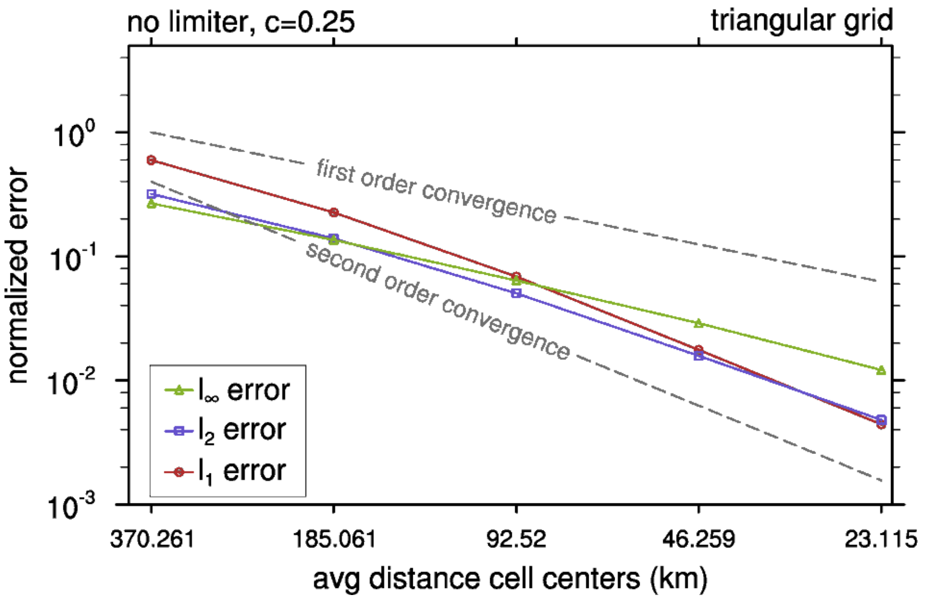
- Uniform flow along northeast direction
- Initial scalar field is a cosine bell centered at the equator
- After 12 days of model integration, cosine bell reaches its initial position
- Analytic solution at every time step = initial condition



Error norms (l_1, l_2, l_∞) are calculated after one complete revolution for different resolutions

Error norms (solid body rotation)

c≈0.25, without limiter



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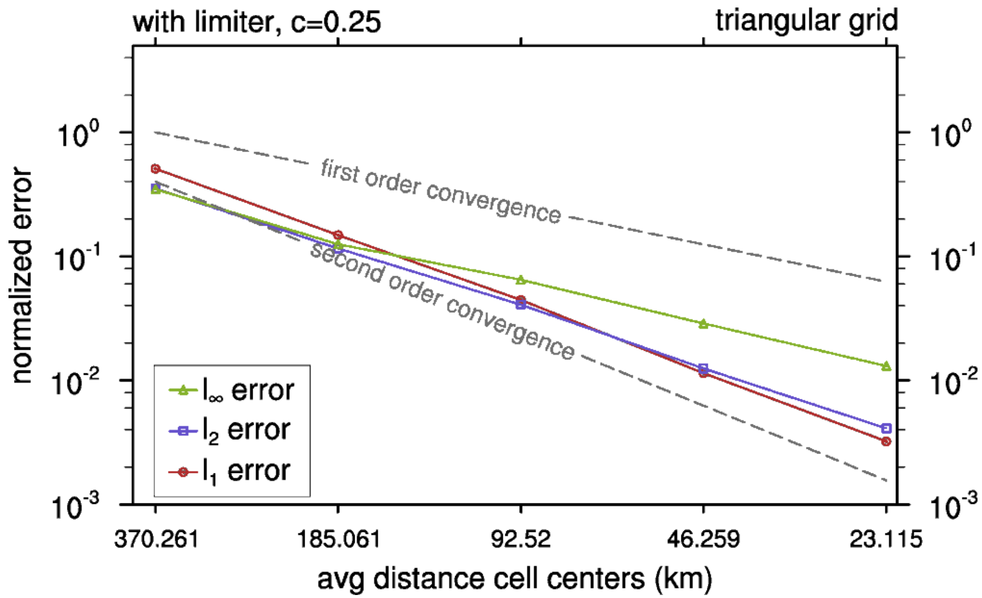
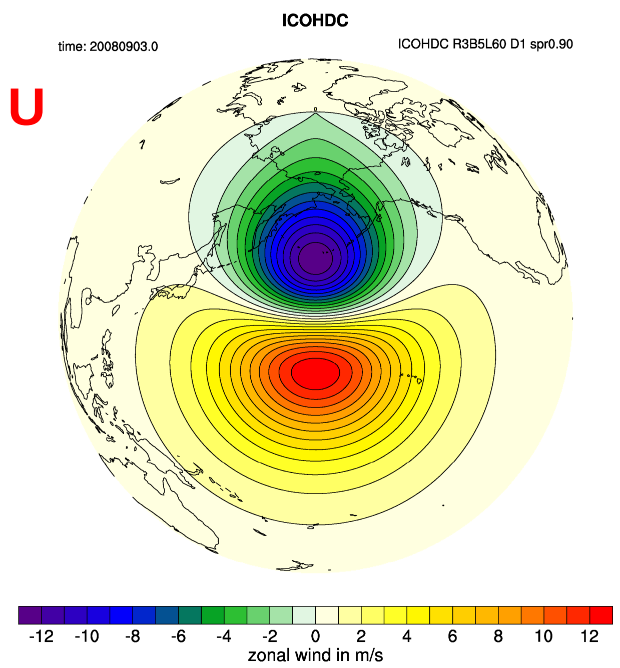
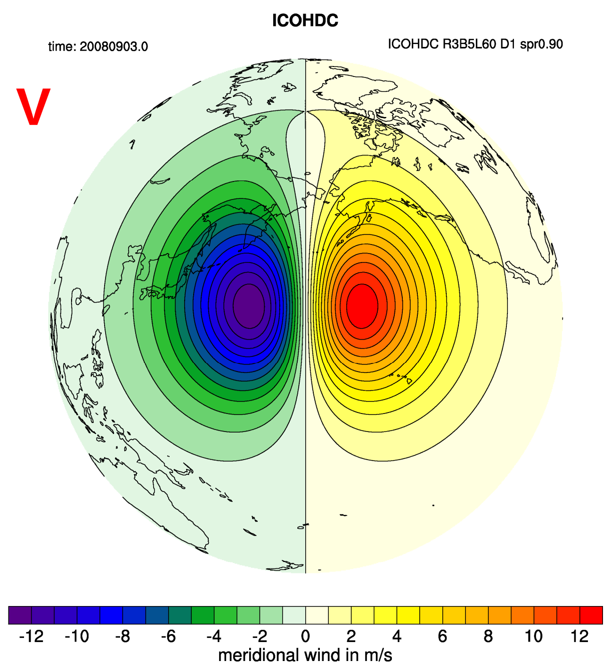


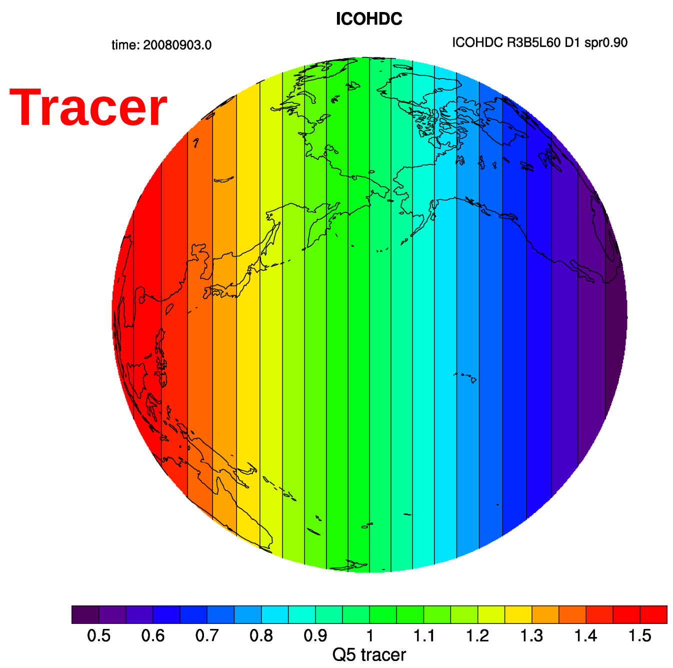
- **l_1 and l_2:** almost 2nd order convergence rate
- **l_∞:** only close to 1st order

Static vortex test case

- Flow is deformational, thus more challenging than solid-body rotation
- Vortex center at (λ, φ)=(180°, 37.5°)
- U, V constant in time
- Comparison against analytical solution



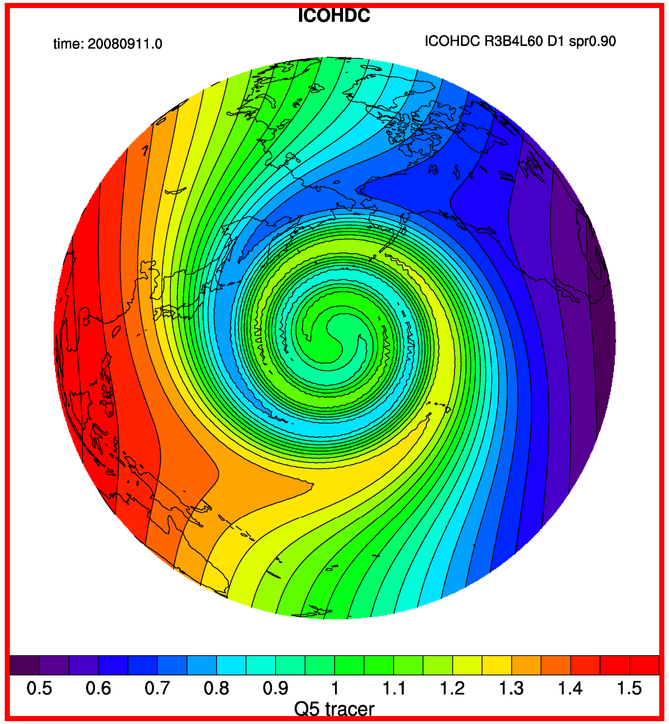




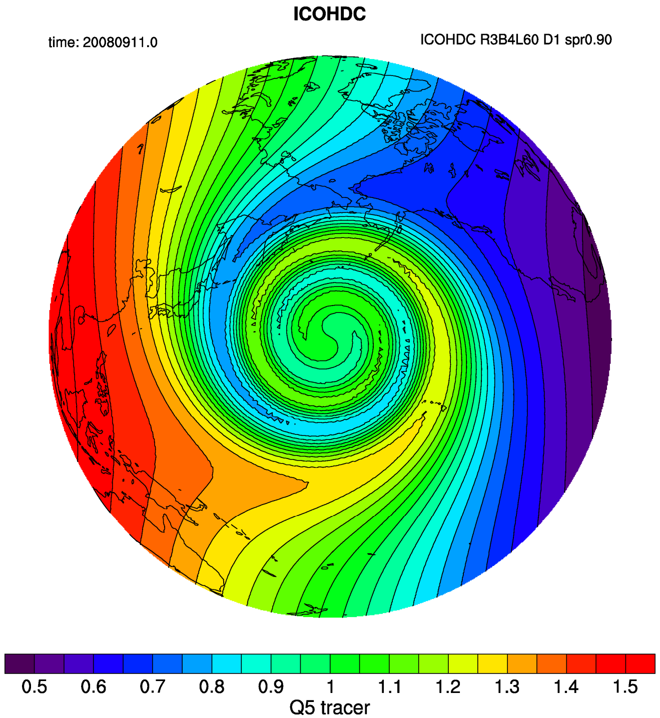
Results after 10 days

- Resolution: R3B4 (≈ 92 km)
- Courant number c ≈ 0.1

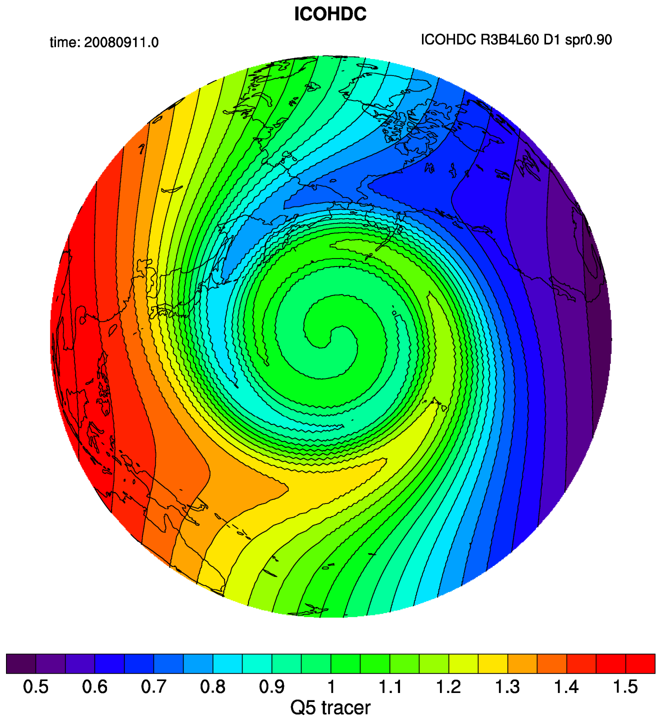
Analytic solution



our scheme (Miura)

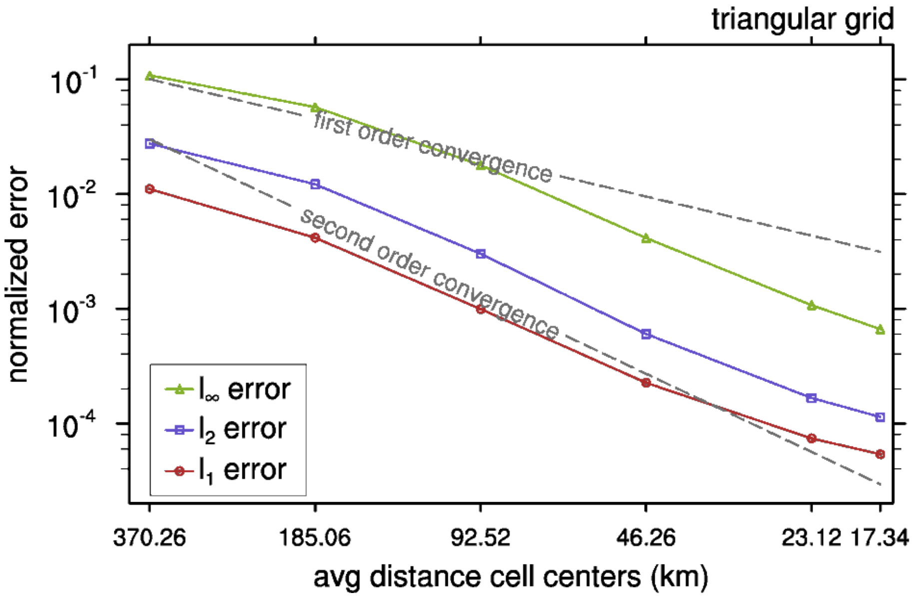


upwind

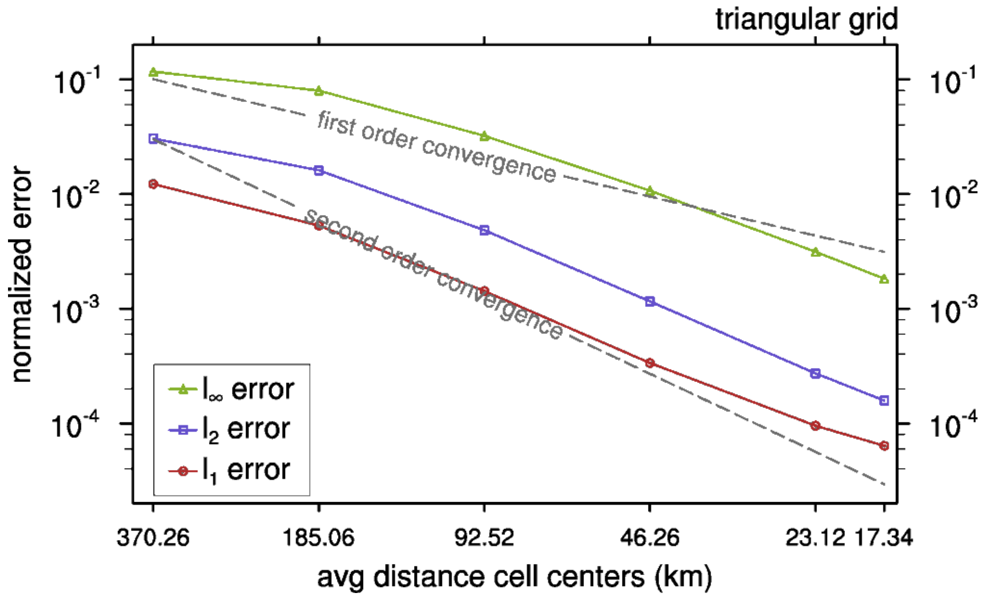


Error norms (static vortex)

c≈0.25, without limiter



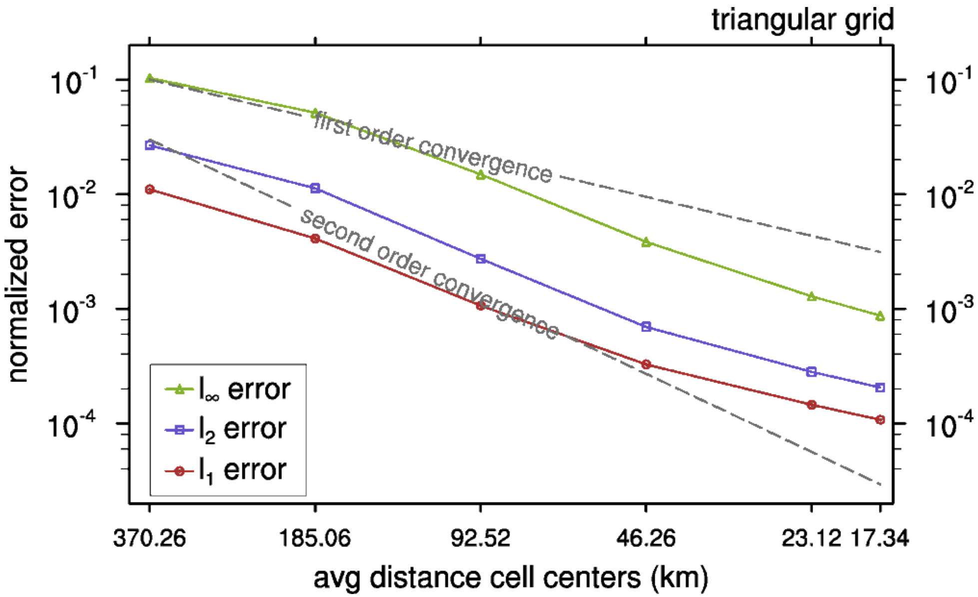
c≈ 0.25 , with limiter



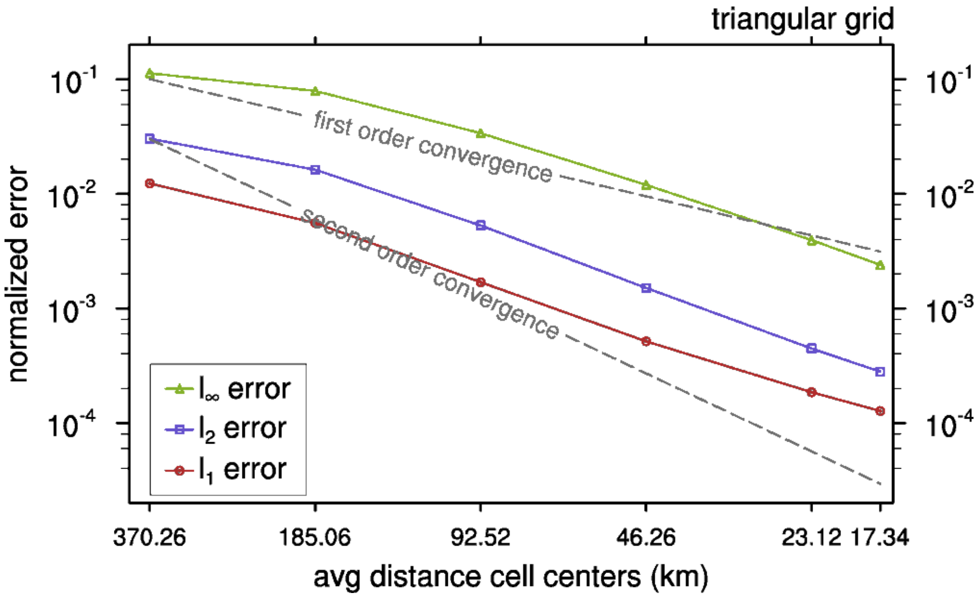
- **no limiter:** almost 2nd order convergence for medium-high resolution, but convergence rate decreases for very high resolution**.**
- **with limiter:** slightly increasing absolute error and slightly decreasing convergence rates

Error norms (static vortex)

c≈0.6, without limiter



c≈0.6 , with limiter



V. Summary and outlook

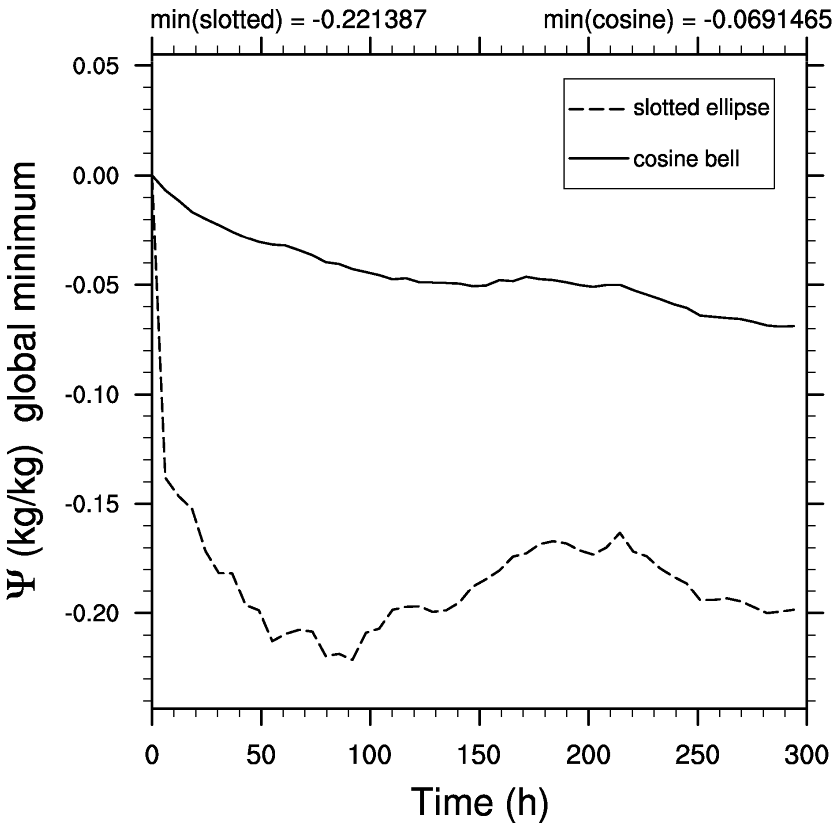
- transport schemes in ICON are still work in progress
- current horizontal advection scheme is similar to the scheme of Miura (2007).
- based on combined semi-Lagrangian finite-Volume thinking (subgrid streamline integration method)
- formally second order accurate
- observed convergence rates are slightly worse than second order (for c≈0.25) and slightly decrease for higher Courant numbers.

Outlook:

- More testing required
- pushing towards third order accuracy
  - Use quadratic instead of linear reconstruction
  - Improve approximation of the upwind area (use reconstructed velocities at vertices)
    - Involves explicit numerical integration of $$\iint_{A_{ik}} \Psi(x, y, t_0)\,\mathrm{d}A$$

Thank you for your attention !!

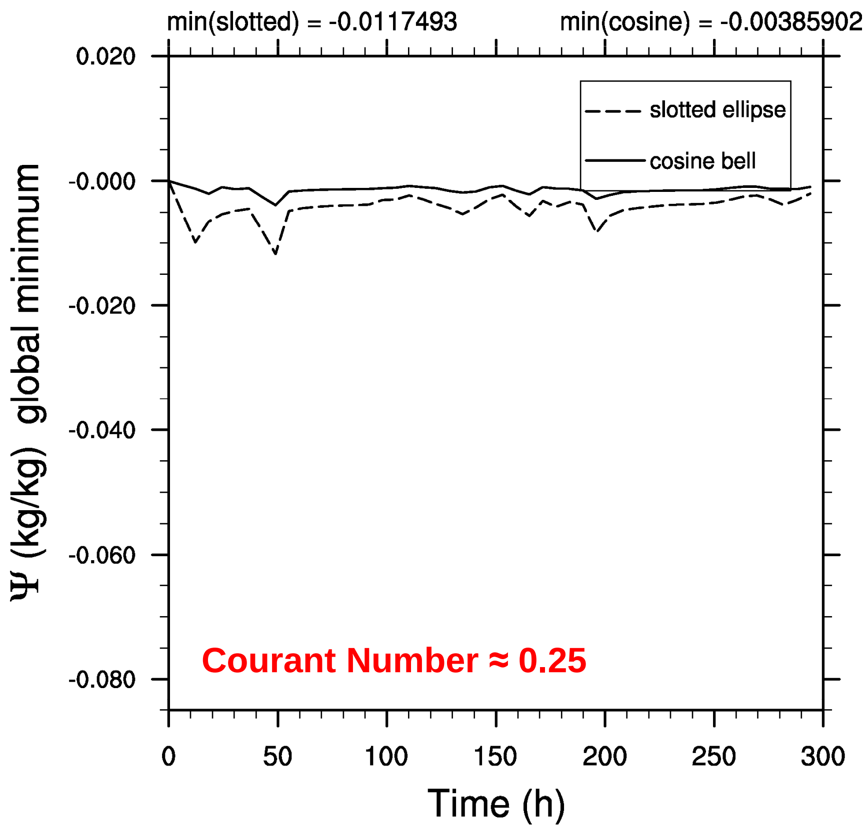
Slope limiting

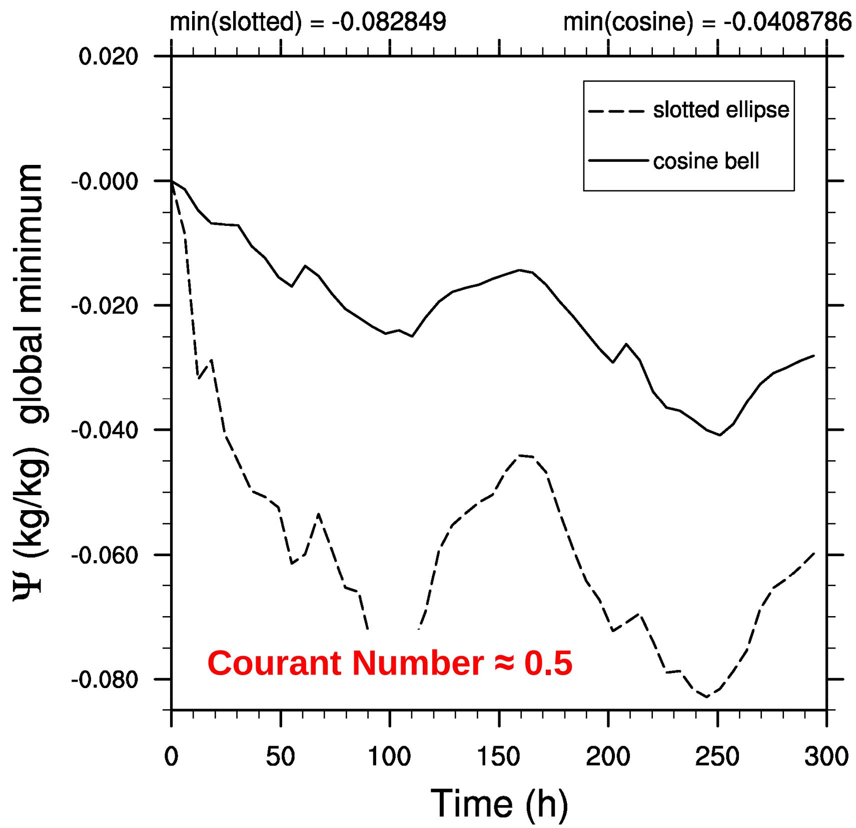


- Initial conditions: 0 <= Ψ <= 1

- Scheme is non-monotonous

Standard Barth-Jespersen limiter





- Not able to fully eliminate over- and undershoots

Modified Limiter