Partially Implicit Time Integration Methods for the Compressible Euler Equations

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SNRWP Workshop Bad Orb

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1 Rosenbrock and Peer methods
   • Motivation
   • Rosenbrock-W-methods
   • Peer methods
   • Order conditions

2 Linear stability theory
   • Linearization of Euler equations
   • A-stability
   • Amplitude and phase properties

3 Numerical tests
   • The 2D compressible Euler equations
   • Rising bubble
   • Flow over mountain
   • Zeppelin test

4 Conclusions and outlook
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   - Motivation
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4 Conclusions and outlook
Cut cell approach with small grid cells
In compressible models occur:

- Energetically relevant slow waves (e.g. advection, Rossby waves)
- Energetically irrelevant fast waves (e.g. sound waves)

In explicit models the fast waves restrict the maximal time step size

One ansatz to overcome this is operator splitting

- Advantages: Every step is cheap, easy to implement, parallelization
- Disadvantages: Still explicit (i.e. only small time steps allowed especially when used together with cut-cells), complicated derivation of order conditions and stability results

Another ansatz is the use of implicit methods

- Advantages: Allows very big time steps, order conditions and stability issues are obvious
- Disadvantages: Requires solution of huge (non-)linear systems of equations, needs efficient (parallel) preconditioners
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Consider PDE discretized in space

\[ y' = f(y) \]

Rosenbrock Method

\[
y_{n+1} = y_n + \sum_{i=1}^{s} b_i k_i, \]

\[
k_i = \tau f \left( y_n + \sum_{j=1}^{i-1} a_{ij} k_j \right) + \Delta t W \sum_{j=1}^{i} \gamma_{ij} k_j, \quad i = 1, ..., s. \]

where \( W \approx J_n = f'(y_n) \).
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**Order conditions**

<table>
<thead>
<tr>
<th>Order $p$</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\sum_{i=1}^{s} b_i = 1$</td>
</tr>
</tbody>
</table>
| 2         | $\sum_{i=2}^{s} b_i a_i = 1/2$
|           | $\sum_{i=2}^{s} b_i d_i = 0$ |
| 3         | $\sum_{i=2}^{s} b_i a_i^2 = 1/3$
|           | $\sum_{i=3}^{s} \sum_{j=2}^{i-1} b_i a_{ij} a_j = 1/6$
|           | $\sum_{i=3}^{s} \sum_{j=2}^{i-1} b_i a_{ij} d_j = 0$
|           | $\sum_{i=3}^{s} \sum_{j=2}^{i-1} b_i \gamma_{ij} a_j = 0$
|           | $\sum_{i=2}^{s} b_i d_i^2 = 0$ |
Rosenbrock-W method based on RK3

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1/3</th>
<th>1/2</th>
<th>0</th>
</tr>
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\[ A \text{-Matrix} \]

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<tr>
<th>( \gamma )</th>
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<td>( \gamma )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-1/4 + 2( \gamma )</td>
<td>1/4 - 3( \gamma )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tbody>
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\[ \Gamma \text{-Matrix} \]
- **Type of approximate Jacobian**
  - Approximate matrix factorization
    \[ f = f_1 + f_2, \quad J = J_1 + J_2, \quad I - \gamma \tau W = (I - \gamma \Delta t J_1)(I - \gamma \Delta t J_2) \]
  - Jacobian from a low order discretization
    \[ f_L \approx f, \quad W = J_L \]
  - Partial Jacobian, split with respect to space or processes
    \[ f = f_1 + f_2, \quad W = J_1 \]
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* Unit interval, 100 grid cells, comparison of a uniformly spaced grid and a uniformly grid cell with one small grid cell $h_{\text{small}} = 1/10000$.

* Uniformly grid cell with one small grid cell $h_{\text{small}} = 1/10000$, one and tenth revolution of the profile, two different limiters
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- Uniformly grid cell with one small grid cell $h_{\text{small}} = 1/10000$, one and tenth revolution of the profile, two different limiters
Peer Method

Write numerical solutions as:

\[ Y_m := \begin{pmatrix} Y_{m1} \\ \vdots \\ Y_{ms} \end{pmatrix} \approx \begin{pmatrix} y(t_m + c_1 \Delta t) \\ \vdots \\ y(t_m + c_s \Delta t) \end{pmatrix} \in \mathbb{R}^{s \times n}, \quad F_m := f(Y_m) \in \mathbb{R}^{s \times n} \]

Runge-Kutta methods (for autonomous systems) read:

\[ Y_m = Y_{m-1,s} + \Delta t AF_m \]

Explicit peer methods are defined by:

\[ Y_{mi} = B_i Y_{m-1} + \Delta t A_i F_{m-1} + \Delta t R_i F_m \]

Performing one Newton step results in the considered class of linearly implicit peer methods:

\[ Y_m(I - h \gamma J)^T = BY_{m-1} + \Delta t AF_{m-1} + \Delta t RF_m + \Delta t GY_{m-1} J^T + \Delta t HY_m J^T \]
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\]
Order conditions $AB(k) = 0$, $\widehat{AB}(k) = 0$, $k \leq s$, can be written in compact matrix form

$$B\mathbb{1} = \mathbb{1},$$

$$A = CV_0D^{-1}V_1^{-1} - B(C - I)V_1D^{-1}V_1^{-1} - RV_0V_1^{-1},$$

$$G = -\Gamma V_0V_1^{-1} - HV_0V_1^{-1}$$

with $\mathbb{1} = (1, \ldots, 1)^T$, $C = \text{diag}(c_1, \ldots, c_s)$, $\Gamma = \gamma I$, $D = \text{diag}(1, 2, \ldots, s)$,

$$V_0 = \begin{pmatrix} 1 & c_1 & \cdots & c_1^{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_s & \cdots & c_s^{s-1} \end{pmatrix}$$

and

$$V_1 = \begin{pmatrix} 1 & c_1 - 1 & \cdots & (c_1 - 1)^{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_s - 1 & \cdots & (c_s - 1)^{s-1} \end{pmatrix}.$$
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In the remainder we will concentrate on second-order methods with $s = 2$ stages. Furthermore we choose $c_s = 1$ so that $Y_{ms} \approx y(t_{m+1})$.

Remaining parameters are $c_1, \gamma, b_{11}, b_{21}, r_{21}$ and $h_{21}$. These will be optimized with respect to good stability properties.
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4 Conclusions and outlook
One-dimensional compressible Euler equations in conservative form:

\[
\begin{align*}
\dot{\rho} &= -\frac{\partial \rho u}{\partial x} \\
\dot{\rho}u &= -\frac{\partial \rho uu}{\partial x} - \frac{\partial p}{\partial x} \\
\dot{\rho}\theta &= -\frac{\partial \rho u\theta}{\partial x} \\
p &= \left(\frac{R\rho\theta}{p_0^\kappa}\right)^{\frac{1}{1-\kappa}}
\end{align*}
\]

Elimination of pressure:

\[
\begin{align*}
\frac{\partial p}{\partial x} &= \frac{\partial p}{\partial \rho\theta} \frac{\partial \rho\theta}{\partial x} \\
\frac{\partial p}{\partial \rho\theta} &= \frac{R}{p_0^\kappa(1-\kappa)} \left(\frac{R\rho\theta}{p_0^\kappa}\right)^{\frac{\kappa}{1-\kappa}} = \frac{1}{\rho\theta(1-\kappa)} \left(\frac{R\rho\theta}{p_0^\kappa}\right)^{\frac{1}{1-\kappa}} = \frac{c_s^2}{\theta}
\end{align*}
\]

with \(c_s\) the speed of sound

\[
c_s := \sqrt{\frac{1}{\rho(1-\kappa)} \left(\frac{R\rho\theta}{p_0^\kappa}\right)^{\frac{1}{1-\kappa}}}
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\]
- Use of product rule for

\[
\frac{\partial \rho uu}{\partial x} = -u^2 \frac{\partial \rho}{\partial x} + 2u \frac{\partial \rho u}{\partial x} \\
\frac{\partial \rho u \theta}{\partial x} = -u \theta \frac{\partial \rho}{\partial x} + \theta \frac{\partial \rho u}{\partial x} + u \frac{\partial \rho \theta}{\partial x}
\]

results in the nonlinear Euler equations in compact form:

\[
\begin{pmatrix}
\dot{\rho} \\
\dot{\rho} u \\
\dot{\rho} \theta
\end{pmatrix}
= - \begin{pmatrix}
0 & 1 & 0 \\
-\frac{\partial \rho}{\partial x} & \frac{\partial \rho u}{\partial x} & \frac{\partial \rho \theta}{\partial x}
\end{pmatrix}
\begin{pmatrix}
\rho_x \\
(\rho u)_x \\
(\rho \theta)_x
\end{pmatrix}
\]

- Linearization by considering the disturbed quantities (e.g. \( \rho' := \rho - \bar{\rho} \)) and dropping all nonlinear terms:

\[
\begin{pmatrix}
\dot{\rho}' \\
(\rho u)_x' \\
\frac{1}{\theta} (\rho \theta)'_x
\end{pmatrix}
= - \begin{pmatrix}
0 & 1 & 0 \\
-\frac{\partial \rho}{\partial x} & \frac{\partial \rho u}{\partial x} & \frac{\partial \rho \theta}{\partial x}
\end{pmatrix}
\begin{pmatrix}
\rho_x' \\
(\rho u)_x' \\
\frac{1}{\theta} (\rho \theta)'_x
\end{pmatrix}
\]

\[
M
\]
Use of product rule for

\[
\frac{\partial \rho uu}{\partial x} = -u^2 \frac{\partial \rho}{\partial x} + 2u \frac{\partial \rho u}{\partial x} \]

\[
\frac{\partial \rho u \theta}{\partial x} = -u \theta \frac{\partial \rho}{\partial x} + \theta \frac{\partial \rho u}{\partial x} + u \frac{\partial \rho \theta}{\partial x}
\]

results in the nonlinear Euler equations in compact form:

\[
\begin{pmatrix}
\dot{\rho} \\
\dot{\rho} u \\
\dot{\rho} \theta
\end{pmatrix} = -
\begin{pmatrix}
0 & 1 & 0 \\
-u^2 & 2u & c_s^2 \theta \\
-u \theta & \theta & u
\end{pmatrix}
\begin{pmatrix}
\rho_x \\
(\rho u)_x \\
(\rho \theta)_x
\end{pmatrix}
\]

Linearization by considering the disturbed quantities (e.g. \( \rho' := \rho - \bar{\rho} \)) and dropping all nonlinear terms:
Use of product rule for

\[ \frac{\partial \rho uu}{\partial x} = -u^2 \frac{\partial \rho}{\partial x} + 2u \frac{\partial \rho u}{\partial x} \]
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results in the nonlinear Euler equations in compact form:

\[
\begin{pmatrix} \dot{\rho} \\ \dot{\rho} u \\ \dot{\rho} \theta \end{pmatrix} = -\begin{pmatrix} 0 & 1 & 0 \\ -u^2 & 2u & \frac{c_s^2}{\theta} \\ -u \theta & \theta & u \end{pmatrix} \begin{pmatrix} \rho_x \\ (\rho u)_x \\ (\rho \theta)_x \end{pmatrix}
\]

Linearization by considering the disturbed quantities (e.g. \( \rho' := \rho - \bar{\rho} \)) and dropping all nonlinear terms:

\[
\begin{pmatrix} \dot{\rho}' \\ (\rho u)_x' \\ \frac{1}{\theta} (\rho \theta)_x' \end{pmatrix} = -\begin{pmatrix} 0 & 1 & 0 \\ -\frac{\bar{u}^2}{u} & 2\frac{\bar{u}}{u} & \frac{c_s^2}{\theta} \\ -\bar{u} & 1 & \frac{1}{\bar{u}} \end{pmatrix} \begin{pmatrix} \rho_x' \\ (\rho u)_x' \\ \frac{1}{\theta} (\rho \theta)_x' \end{pmatrix}
\]

\[ M \]
To save storage and gain computational efficiency we make two simplifications for the Jacobian $J$:

- Use Jacobian of the advection form of the Euler equations
- Use first-order upwind scheme for spatial discretization

Use $\bar{\rho}u' \approx (\rho u)' - \bar{u}\rho$ instead of $(\rho u)'$, i.e. use:

$$
\tilde{M} := - \begin{pmatrix}
1 & 0 & 0 \\
-\bar{u} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
-\bar{u}^2 & 2\bar{u} & c_s^2 \\
-\bar{u} & 1 & \bar{u}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\bar{u} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
= - \begin{pmatrix}
\bar{u} & 1 & 0 \\
0 & \bar{u} & c_s^2 \\
0 & 1 & \bar{u}
\end{pmatrix}
$$

It holds:

$$
\begin{pmatrix}
\dot{\rho}' \\
\bar{\rho}u' \\
\frac{1}{\theta}(\rho\theta)'
\end{pmatrix}
= - \begin{pmatrix}
\bar{u} & 1 & 0 \\
0 & \bar{u} & c_s^2 \\
0 & 1 & \bar{u}
\end{pmatrix}
\begin{pmatrix}
\rho_x' \\
(\bar{\rho}u')_x \\
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\end{pmatrix}
$$
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$$\tilde{M} := -\begin{pmatrix} 1 & 0 & 0 \\ -\bar{u} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -\bar{u}^2 & 2\bar{u} & c_s^2 \\ -\bar{u} & 1 & \bar{u} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \bar{u} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -\begin{pmatrix} \bar{u} & 1 & 0 \\ 0 & \bar{u} & c_s^2 \\ 0 & 1 & \bar{u} \end{pmatrix}$$

It holds:

$$\begin{pmatrix} \dot{\rho}' \\ \rho' \dot{u}' \\ \frac{1}{\theta} (\rho' \dot{\theta})' \end{pmatrix} = -\begin{pmatrix} \bar{u} & 1 & 0 \\ 0 & \bar{u} & c_s^2 \\ 0 & 1 & \bar{u} \end{pmatrix} \begin{pmatrix} \rho_x' \\ (\rho' u_x)' \\ \frac{1}{\theta} (\rho' \theta_x)' \end{pmatrix}$$
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\begin{pmatrix}
0 & 1 & 0 \\
-\bar{u}^2 & 2\bar{u} & c_s^2 \\
-\bar{u} & 1 & \bar{u}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\bar{u} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
= -\begin{pmatrix}
\bar{u} & 1 & 0 \\
0 & \bar{u} & c_s^2 \\
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\end{pmatrix}
$$

It holds:

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\begin{pmatrix}
\dot{\rho}' \\
\dot{\bar{\rho}}u' \\
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\end{pmatrix}
= -\begin{pmatrix}
\bar{u} & 1 & 0 \\
0 & \bar{u} & c_s^2 \\
0 & 1 & \bar{u}
\end{pmatrix}
\begin{pmatrix}
\rho_x' \\
(\bar{\rho}u')_x \\
\frac{1}{\bar{\theta}}(\rho\dot{\theta})'_{xx}
\end{pmatrix}
$$
To save storage and gain computational efficiency we make two simplifications for the Jacobian $J$:

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$$

It holds:

$$
\begin{pmatrix} \frac{\rho'}{\rho} \\ \frac{\bar{\rho}u'}{\rho} \\ \frac{1}{\theta}(\rho\theta)' \end{pmatrix} = -\begin{pmatrix} \bar{u} & 1 & 0 \\ 0 & \bar{u} & c_s^2 \\ 0 & 1 & \bar{u} \end{pmatrix} \begin{pmatrix} \frac{\rho'}{\rho} \\ (\bar{\rho}u')_x \\ \frac{1}{\theta}(\rho\theta)'_x \end{pmatrix}
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$$
Variables are defined on a staggered grid

\[ \rho u \bigg|_{j-1/2} = \rho u \bigg|_{j+1/2} = \rho u \]

For investigation of spatial discretizations perform von Neumann stability analysis, e.g. it holds:

\[
\rho u(t, x_{j+1/2}) = \rho u(t)e^{ikx_{j+1/2}}
\]

\[ \Rightarrow \frac{\partial \rho u}{\partial x} \bigg|_{(t,x_j)} = \rho u(t) \frac{e^{ikx_j}}{\Delta x} \left( e^{\frac{ik\Delta x}{2}} - e^{-\frac{ik\Delta x}{2}} \right) \]

Three spatial discretizations appear:

\[ D_1 = \frac{1}{\Delta x} \left( 1 - e^{-ik\Delta x} \right) \]

\[ D_2 = \frac{1}{\Delta x} \left( e^{\frac{ik\Delta x}{2}} - e^{-\frac{ik\Delta x}{2}} \right) \]

\[ D_3 = \frac{1}{6\Delta x} \left( 2e^{ik\Delta x} + 3 - 6e^{-ik\Delta x} + e^{-2ik\Delta x} \right) \]
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\[
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\]
• Variables are defined on a staggered grid

\[ \begin{array}{ccccccc}
  & & j-1 & j-1/2 & j & j+1/2 & j+1 \\
\cdots\cdots & \rho u & \rho \theta & \rho u & \rho \theta & \rho u & \rho \theta & \rho u & \rho \theta & \rho u \\
\cdots\cdots & & & & & & & & & \\
\end{array} \]

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Using these operators results in the ODE:

\[
\begin{pmatrix}
\dot{\rho}' \\
(\dot{\rho}u)' \\
\frac{1}{\theta}(\dot{\rho}\theta)'
\end{pmatrix}
= - 
\begin{pmatrix}
0 & D_2 & 0 \\
-\bar{u}^2D_3 & 2\bar{u}D_3 & c_s^2D_2 \\
-\bar{u}D_3 & D_2 & \frac{2}{\bar{u}}D_3
\end{pmatrix}
\begin{pmatrix}
\rho' \\
(\rho u)' \\
\frac{1}{\theta}(\rho\theta)'
\end{pmatrix}
\]

For the Jacobian we instead use the matrix which belongs to:

\[
\begin{pmatrix}
\dot{\rho}' \\
\dot{\rho}u' \\
\frac{1}{\theta}(\dot{\rho}\theta)'
\end{pmatrix}
= - 
\begin{pmatrix}
\bar{u}D_1 & D_2 & 0 \\
0 & \bar{u}D_1 & c_s^2D_2 \\
0 & D_2 & \frac{2}{\bar{u}}D_1
\end{pmatrix}
\begin{pmatrix}
\rho' \\
\rho u' \\
\frac{1}{\theta}(\rho\theta)'
\end{pmatrix}
\]

Remark: While \( M \) and \( \tilde{M} \) are similar the matrices

\[
A := 
\begin{pmatrix}
0 & D_2 & 0 \\
-\bar{u}^2D_3 & 2\bar{u}D_3 & c_s^2D_2 \\
-\bar{u}D_3 & D_2 & \frac{2}{\bar{u}}D_3
\end{pmatrix}
and
\tilde{A} := 
\begin{pmatrix}
\bar{u}D_1 & D_2 & 0 \\
0 & \bar{u}D_1 & c_s^2D_2 \\
0 & D_2 & \frac{2}{\bar{u}}D_1
\end{pmatrix}
\]

are not similar.
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\[
\begin{pmatrix}
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-\bar{u}D_3 & D_2 & \frac{\bar{u}}{u}D_3
\end{pmatrix}
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0 & \bar{u}D_1 & c_s^2D_2 \\
0 & D_2 & \frac{\bar{u}}{u}D_1
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\begin{pmatrix}
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\quad \text{and} \quad
\tilde{A} := \begin{pmatrix}
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0 & \bar{u}D_1 & c_s^2D_2 \\
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-\overline{u} D_3 & D_2 & \overline{u} D_3
\end{pmatrix}
\begin{pmatrix}
\rho' \\
(\rho u)' \\
\frac{1}{\theta} (\rho \theta)'
\end{pmatrix}
\]

For the Jacobian we instead use the matrix which belongs to:

\[
\begin{pmatrix}
\dot{\rho}' \\
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\frac{1}{\theta} (\bar{\rho} \dot{\theta})'
\end{pmatrix}
= - \begin{pmatrix}
\overline{u} D_1 & D_2 & 0 \\
0 & \overline{u} D_1 & c_s^2 D_2 \\
0 & D_2 & \overline{u} D_1
\end{pmatrix}
\begin{pmatrix}
\rho' \\
\bar{\rho} u' \\
\frac{1}{\theta} (\bar{\rho} \dot{\theta})'
\end{pmatrix}
\]

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A := \begin{pmatrix}
0 & D_2 & 0 \\
-\overline{u}^2 D_3 & 2\overline{u} D_3 & c_s^2 D_2 \\
-\overline{u} D_3 & D_2 & \overline{u} D_3
\end{pmatrix}
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0 & \overline{u} D_1 & c_s^2 D_2 \\
0 & D_2 & \overline{u} D_1
\end{pmatrix}
\]

are not similar.
Eigenvalues of correct and simplified Jacobian
Application of the peer method to the Dahlquist test equation

\[ \dot{y} = \lambda y \]

leads to:

\[ (1 - \Delta t \gamma J) Y_m = B Y_{m-1} + \Delta t A \lambda Y_{m-1} + \Delta t R \lambda Y_m + \Delta t G J Y_{m-1} + \Delta t H J Y_m \]

With notations \( z := \Delta t \lambda \) and \( \tilde{z} := \Delta t J \) it holds:

\[ Y_m = (I - z R - \tilde{z} (\gamma I + H))^{-1} (B + z A + \tilde{z} G) Y_{m-1} \]

Side conditions for optimization are

- A-stability in common sense, i.e. for \( \tilde{z} = z \)
- A-stability for simplified Jacobian, i.e. for \( \text{Re}\tilde{z} = 2.5 \text{Re}z, \text{Im}\tilde{z} = \text{Im}z \)
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Stability regions for exact and simplified Jacobian
The analytical solution of the Dahlquist test equation satisfies

\[ y(t_m) = e^{\tilde{y}} y(t_{m-1}) = e^{Rez} e^{iImz} y(t_{m-1}), \]

i.e. the analytical solution has

- the amplification factor \( e^{Rez} \)
- the relative phase speed \( 1 \)

Let \( \lambda \) be an eigenvalue of the amplification matrix

\[ (I - zR - \tilde{z}(\gamma I + H))^{-1}(B + zA + \tilde{z}G) \]

of a peer method applied to the Dahlquist test equation

- The amplification factor is \( |\lambda| \)
- The relative phase speed is \( \frac{\arctan \frac{Im\lambda}{Re\lambda}}{Im\lambda} \)

Optimization goal are good amplitude and phase errors for the case \( z = (-0.05 + i)Imz \) (i.e. eigenvalues of advection and acoustics) when using the simplified Jacobian (i.e. for \( Re\tilde{z} = 2.5Rez, Im\tilde{z} = Imz \))
The analytical solution of the Dahlquist test equation satisfies

\[ y(t_m) = e^{z} y(t_{m-1}) = e^{\Re z} e^{i \Im z} y(t_{m-1}), \]

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Let \( \lambda \) be an eigenvalue of the amplification matrix

\[ (I - zR - \tilde{z}(\gamma I + H))^{-1}(B + zA + \tilde{z}G) \]

of a peer method applied to the Dahlquist test equation

- The amplification factor is \( |\lambda| \)
- The relative phase speed is \( \frac{\arctan \frac{Im\lambda}{Re\lambda}}{Im\lambda} \)

Optimization goal are good amplitude and phase errors for the case \( z = (-0.05 + i)Im\tilde{z} \) (i.e. eigenvalues of advection and acoustics) when using the simplified Jacobian (i.e. for \( Re\tilde{z} = 2.5Re\tilde{z}, Im\tilde{z} = Imz \))
The analytical solution of the Dahlquist test equation satisfies

\[ y(t_m) = e^{\tilde{z}}y(t_{m-1}) = e^{\Re z} e^{i\Im z} y(t_{m-1}), \]

i.e. the analytical solution has

- the amplification factor \( e^{\Re z} \)
- the relative phase speed \( 1 \)

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Amplitude and phase for the simplified Jacobian
1. Rosenbrock and Peer methods
   - Motivation
   - Rosenbrock-W-methods
   - Peer methods
   - Order conditions

2. Linear stability theory
   - Linearization of Euler equations
   - A-stability
   - Amplitude and phase properties

3. Numerical tests
   - The 2D compressible Euler equations
   - Rising bubble
   - Flow over mountain
   - Zeppelin test

4. Conclusions and outlook
\[
\begin{align*}
\frac{\partial \rho}{\partial t} &= -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial z} \\
\frac{\partial \rho u}{\partial t} &= -\frac{\partial \rho uu}{\partial x} - \frac{\partial \rho uw}{\partial z} - \frac{R}{1 - \kappa} \frac{\partial \rho \theta}{\partial x} \\
\frac{\partial \rho w}{\partial t} &= -\frac{\partial \rho uw}{\partial x} - \frac{\partial \rho ww}{\partial z} - \frac{R}{1 - \kappa} \frac{\partial \rho \theta}{\partial z} - \rho g \\
\frac{\partial \rho \theta}{\partial t} &= -\frac{\partial \rho u \theta}{\partial x} - \frac{\partial \rho w \theta}{\partial z} \\
\pi &= \left( \frac{R \rho \theta}{p_0} \right)^{\frac{\kappa}{1 - \kappa}}
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\end{align*}
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<table>
<thead>
<tr>
<th></th>
<th>correct Jacobian</th>
<th>simplified Jacobian</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1D</td>
<td>$3D_2 + 4D_3 = 22$</td>
<td>$3D_2 + 3D_1 = 12$</td>
<td>55%</td>
</tr>
<tr>
<td>2D</td>
<td>$6D_2 + 14D_3 = 68$</td>
<td>$6D_2 + 8D_1 = 28$</td>
<td>41%</td>
</tr>
<tr>
<td>3D</td>
<td>$9D_2 + 30D_3 = 138$</td>
<td>$9D_2 + 15D_1 = 48$</td>
<td>35%</td>
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Rising bubble
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Conclusions

- Development of a linearly implicit two-stage peer method which
  - is second-order independently of the Jacobian
  - is A-stable in the common sense and for the simplified Jacobian
  - has acceptable amplitude and phase errors
- Despite of the large CFL numbers the solutions of the linearly implicit peer method are as good as the solutions computed with the explicit method with tiny time steps
- Only exception is the transported rising bubble where the impact of damping and phase errors is visible, but
  - the explicit method is a three-stage method, there is no explicit two-stage method which is stable with the time steps used in the first test
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Outlook

- Determination of the practical speed-up when using the simplified Jacobian instead of the correct one
- Mixing of linearly implicit and explicit peer methods:
  - Use of full Jacobian in regions where orography results in cut-cells
  - In free regions without cut-cells only the parts of the Jacobian which come from acoustics have non-zeros entries
- Such a peer method should
  - compute with time step sizes restricted only by the CFL condition of the underlying explicit method in the free regions
  - produce as good results as the split-explicit peer method
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Danke für eure/Ihre Aufmerksamkeit!