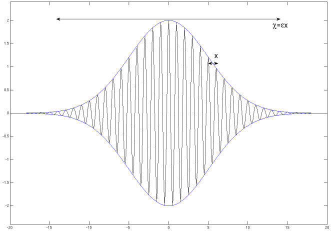
Validation of an extended non-linear WKB theory for gravity wave propagation



9th Int. SRNWP-Workshop on Nonhydrostatic Modelling
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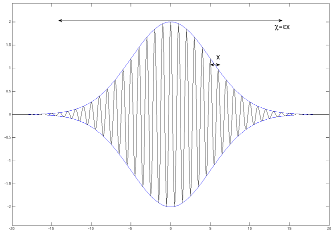
WKB as parametrization scheme for GWs

- Simple example why we need parametrizations...
- Full model must resolve waves
  $\Rightarrow$ CPU time $\sim$ 3 h
- WKB model does not need to resolve waves
  $\Rightarrow$ CPU time $\sim$ 10 s



Contents

Full model: Equations

Sound-proof pseudo-incompressible equations [Durran, 1989]

The pseudo-incompressible equations (scaled, conservative)

Background: $\bar{\rho}(z), \bar{\theta}(z), \bar{p}(z)$

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g, \quad \bar{\rho}\bar{\theta} = \bar{p}^{1/\gamma}$$

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Variables: $\rho^*, \mathbf{u}, \pi$

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Exner pressure

$$\pi = \left(\frac{p}{p_0}\right)^{\kappa}, \quad \kappa = \frac{R}{c_p}, \quad \gamma = \frac{c_p}{c_V}$$

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Prognostic and diagnostic equations:

$$\rho^*_t + \nabla \cdot (\rho^* \mathbf{u}) = 0$$

$$(\rho^* \mathbf{u})_t + \nabla \cdot (\rho^* \mathbf{u} \circ \mathbf{u}) + \frac{1}{\mathrm{Ma}^2 \kappa} \bar{\rho}\bar{\theta} \nabla \pi' = \frac{1}{\mathrm{Fr}^2} \rho^* \frac{\theta'}{\bar{\theta}} \mathbf{k} + \frac{1}{\mathrm{Re}} Visc$$

$$\nabla \cdot (\bar{\rho}\bar{\theta} \mathbf{u}) = Heating$$

Full model: Implementation

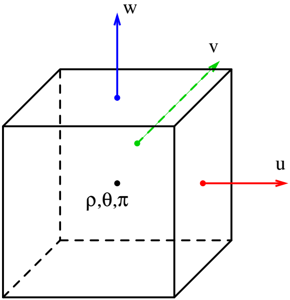
Conserving mass and momentum with FVM

Spatial Discretisation

- Data structure: C-grid
- Conservative treatement of mass and momentum transport

$$\nabla\cdot \quad \to \quad \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x} + \dots$$

$$f_{i+1/2} = \begin{cases} \text{MUSCL: 2nd order upwind} \\ \text{ALDM: implicit LES} \end{cases}$$

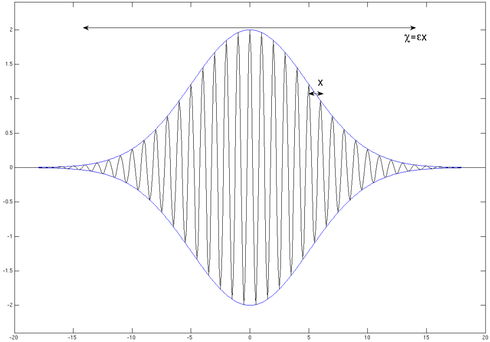


WKB: Equations

A theory for wave packet propagation

WKB - A reduced model for wave propagation

- Wentzel, Kramers and Brillouin for Schrördinger eq. (1926)
- non-linear extension based on multi-scale asymptotics (Achatz, Klein, Senf 2010)
- Asymptotic Ansatz:



WKB - A reduced model for wave propagation

- Asymptotic Ansatz –> Euler equations:

$$\tilde{u} = U_0^{(0)} + \mathcal{R}\left[U_1^{(0)} \exp(i\frac{\phi}{\varepsilon})\right] + \varepsilon\left[U_0^{(1)} + \mathcal{R}\sum_{\alpha=1}^{\infty} U_\alpha^{(1)} \exp(i\alpha\frac{\phi}{\varepsilon})\right] + \mathcal{O}(\varepsilon^2)$$

- with
  $U = U(\tau, \chi, \zeta), \quad \tau = \varepsilon t, \quad \chi = \varepsilon x, \quad \zeta = \varepsilon z, \quad \varepsilon = L/H_\theta$

Linear theory (classical): transport of wave properties

- Dispersionrelation:

$$\hat{\omega}^2 = N^2 \frac{k^2}{k^2 + m^2}$$

- Transport along rays:

$$\frac{\mathrm{D}k}{\mathrm{D}\tau} = 0 \qquad \text{with} \qquad \frac{\mathrm{D}}{\mathrm{D}\tau} = \frac{\partial}{\partial \tau} + \mathbf{c}_g \cdot \nabla_{\chi,\zeta}$$

$$\frac{\mathrm{D}m}{\mathrm{D}\tau} = -k \frac{\partial U_0^{(0)}}{\partial \zeta} \qquad \text{and} \qquad \mathbf{c}_g = \left(U_0^{(0)} + \frac{\partial \hat{\omega}}{\partial k}, \frac{\partial \hat{\omega}}{\partial m}\right)$$

- Transport of wave action:

$$\frac{\partial}{\partial \tau}\left(\frac{\tilde{E}}{\hat{\omega}}\right) + \nabla \cdot \left(\mathbf{c}_g \frac{\tilde{E}}{\hat{\omega}}\right) = 0$$

Non-linear theory –> mean flow & higher harmonics

- OLD: Change of mean flow due to GW:

$$\Rightarrow U_0^{(0)}(\tau)$$

- NEW: Second harmonics can be calculated

$$\Rightarrow U_2^{(1)} \exp(\frac{2i\phi}{\varepsilon})$$

- NEW: No further harmonics

$$\Rightarrow U_\alpha^{(1)} = 0, \ \alpha > 2$$

WKB: Implementation

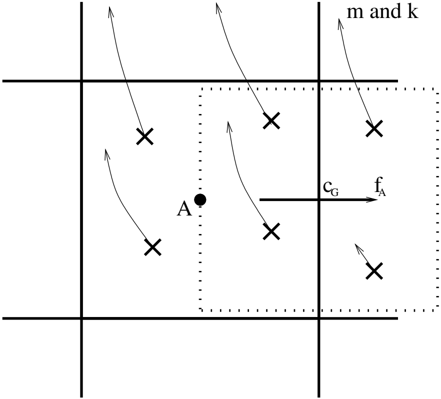
Euler-Lagrangian transport

Euler-Lagrangian transport

$$\frac{\mathrm{D}k}{\mathrm{D}\tau} = 0$$

$$\frac{\mathrm{D}m}{\mathrm{D}\tau} = -k\frac{\partial U_0^{(0)}}{\partial \zeta}$$

$$\frac{\partial A}{\partial \tau} + \nabla \cdot (\mathbf{c}_g A) = 0$$



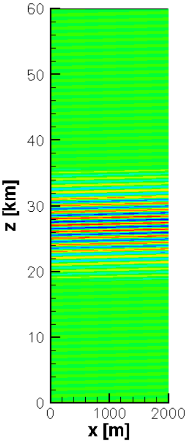
Full vs. WKB model

Using a 1D gravity wave packet

Numerical Validation of non-linear WKB

Test case data:

Domain:	$1.2\,\mathrm{km} \times 60\,\mathrm{km}$
Grid:	64×800
Atmosphere:	isothermal, 300 K
Wave:	$\lambda_x = 1.2\,\mathrm{km},\ \lambda_z = 1.0\,\mathrm{km}$
z-Shape:	Gaussian, $\sigma_z = 5\,\mathrm{km}$
x-Shape:	const height $\Rightarrow$ 1D-wave packet
Amplitude:	$a_0 = 0.1,\ \max(\theta_0') = 0.31\,\mathrm{K}$



Mean flow

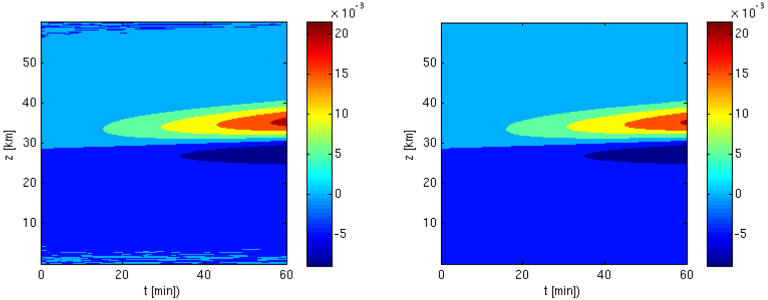


Figure: Hovmöller diagram of GW-induced mean flow for the full model (left) and the WKB model (right)

Wave number 1

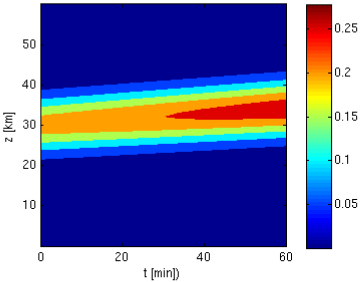


Figure: Hovmöller diagram of wave number m = 1 zonal velocity for the full model (left) and the WKB model (right)

Wave number 2

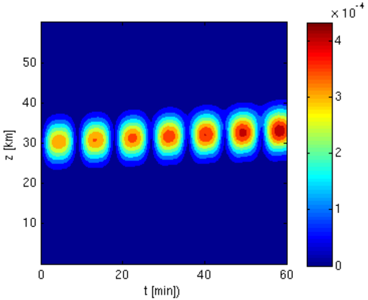


Figure: Hovmöller diagram of wave number $m = 2$ zonal velocity for the full model (left) and the WKB model (right)

Wave number 3

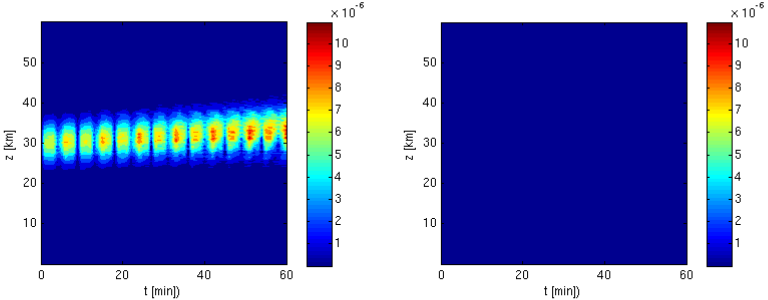


Figure: Hovmöller diagram of wave number m = 3 zonal velocity for the full model (left) and the WKB model (right)

Wave number 2 and 3: Oscillatory amplitude

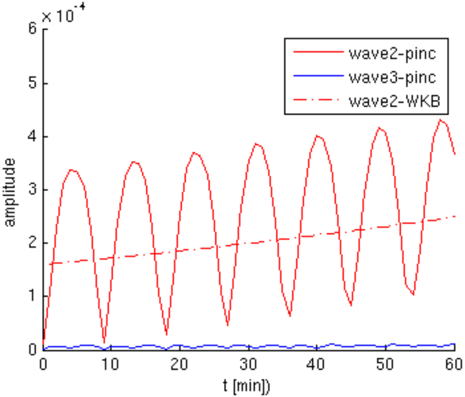


Figure: Amplitudes of wave 2 and 3 of the zonal velocity.

Summary and Outlook

pincFloit: Full model

- Pseudo-incompressible equations
- Implementation

WKB: Reduced model

- review WKB for GW
- Implementation

Validation

- Full model <==> WKB model

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Thank you for your attention!