A Nonhydrostatic Unstructured-Mesh Soundproof Model for Simulation of Internal Gravity Waves

Piotr Smolarkiewicz, National Center for Atmospheric Research*, USA;
Joanna Szmelter, Loughborough University, United Kingdom

The threefold aim of this talk:

a) highlight the progress with the development of a nonhydrostatic, soundproof, unstructured-mesh model for atmospheric flows;

b) assess the accuracy of unstructured-mesh discretization relative to established structured-grid methods for wave dynamics;

c) address relative merits of anelastic and pseudo-incompressible PDEs ➔ implications for soundproof versus fully compressible PDEs

***The National Center for Atmospheric Research is supported by the National Science Foundation**

Unstructured-mesh framework for atmospheric flows

Smolarkiewicz ↻ Szmelter, pubs in *JCP, IJNMF,Comp. Fluids*, 2005-2011

- Differential manifolds formulation $\dfrac{\partial G\boldsymbol{\Phi}}{\partial t} + \nabla \cdot (\mathrm{V}\boldsymbol{\Phi}) = G\boldsymbol{\mathcal{R}}\ ,\mathrm{V}(\mathrm{x},t) := G\dot{\mathrm{x}}$

- Finite-volume NFT numerics with a fully unstructured spatial discretization, heritage of EULAG and its predecessors

$$\boldsymbol{\Phi}_i^{n+1} = \mathcal{A}_i(\boldsymbol{\Phi}^n + 0.5\delta t\,\boldsymbol{\mathcal{R}}^n,\ \mathrm{V}^{n+1/2}, G) + 0.5\delta t\,\boldsymbol{\mathcal{R}}_i^{n+1}$$

- Focus (so far) on wave phenomena across a range of scales and Mach, Froude & Rossby numbers

The median-dual edge-based discretisation

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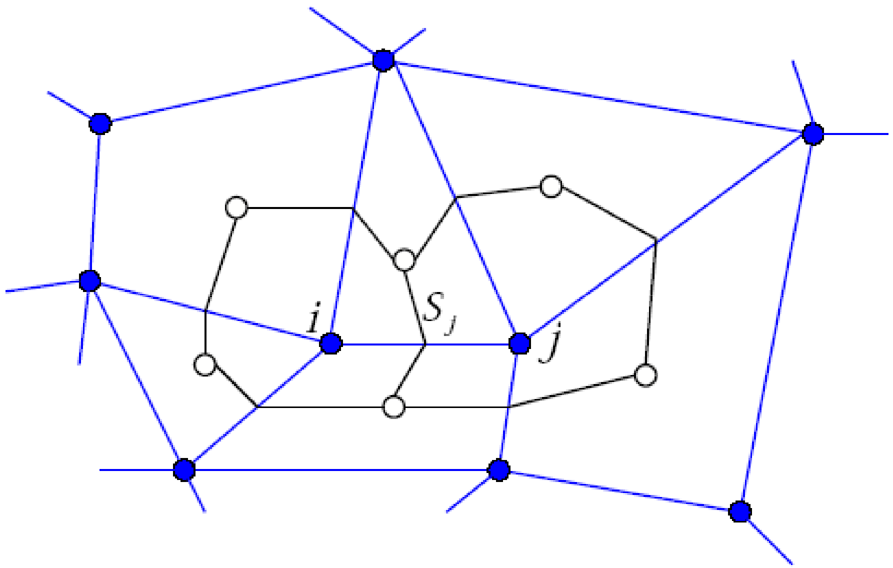
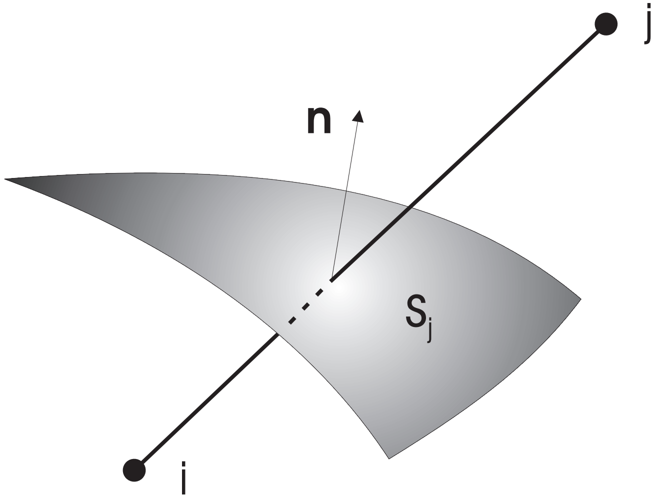
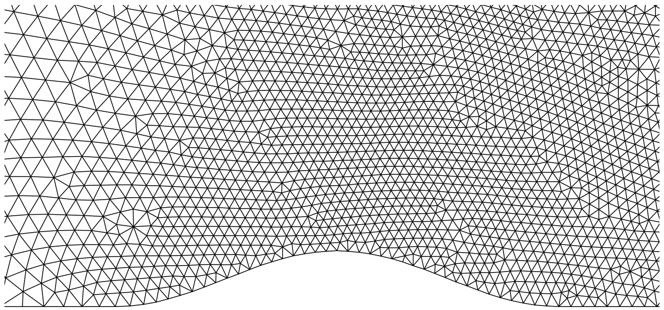
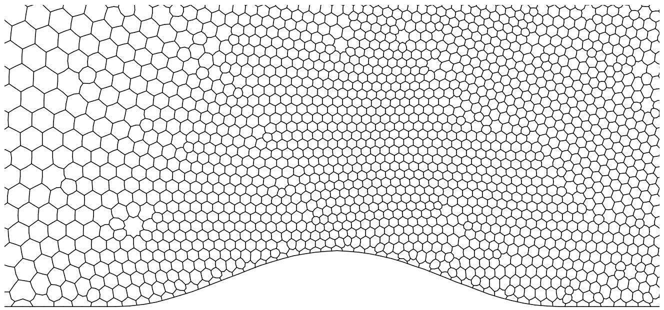






Edges



Dual mesh, finite volumes

Nonhydrostatic Boussinesq mountain wave

Szmelter & Smolarkiewicz , *Comp. Fluids*, 2011

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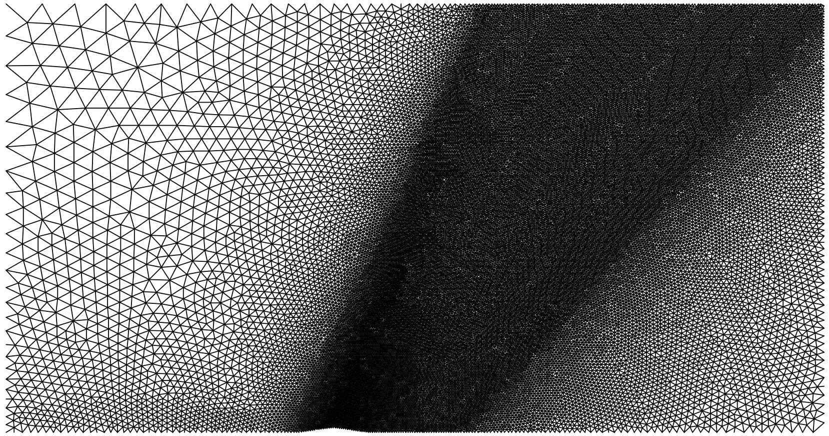
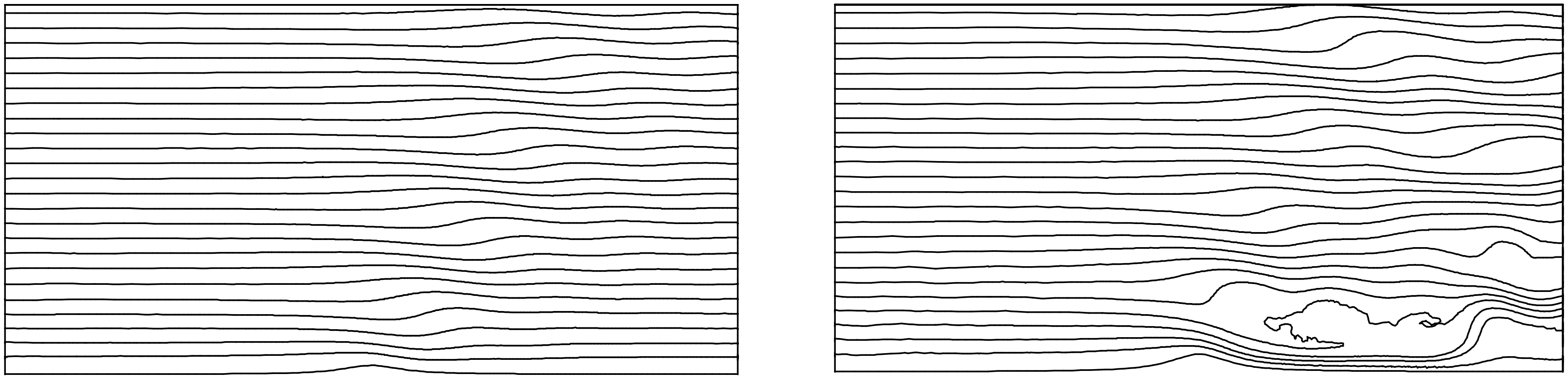
$$\nabla \bullet (\mathbf{V}\rho_o) = 0\ ,$$

$$\frac{\partial \rho_o V^I}{\partial t} + \nabla \bullet (\mathbf{V}\rho_o V^I) = -\rho_o \frac{\partial \tilde{p}}{\partial x^I} + g\rho_o \frac{\theta'}{\theta_o}\delta_{I2}$$

$$\frac{\partial \rho_o \theta}{\partial t} + \nabla \bullet (\mathbf{V}\rho_o \theta) = 0\ .$$





$Fr \lesssim 2$ $\quad$ $NL/U_o = 2.4$ $\quad$ $Fr \lesssim 1$

Comparison with the EULAG's results and the linear theories (Smith 1979, Durran 2003):
3% in wavelength; 8% in propagation angle; wave amplitude loss 7% over 7 wavelenghts

A global hydrostatic sound-proof model

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$$\frac{\partial G\mathcal{D}}{\partial t} + \nabla \cdot (G\mathbf{v}^*\mathcal{D}) = 0 \,,$$

$$\frac{\partial GQ_x}{\partial t} + \nabla \cdot (G\mathbf{v}^*Q_x) = G\left(-\frac{1}{h_x}\mathcal{D}\frac{\partial M}{\partial x} + fQ_y - \frac{1}{G\mathcal{D}}\frac{\partial h_x}{\partial y}Q_xQ_y\right) \,,$$

$$\frac{\partial GQ_y}{\partial t} + \nabla \cdot (G\mathbf{v}^*Q_y) = G\left(-\frac{1}{h_y}\mathcal{D}\frac{\partial M}{\partial y} - fQ_x + \frac{1}{G\mathcal{D}}\frac{\partial h_x}{\partial y}Q_x^2\right) \,,$$

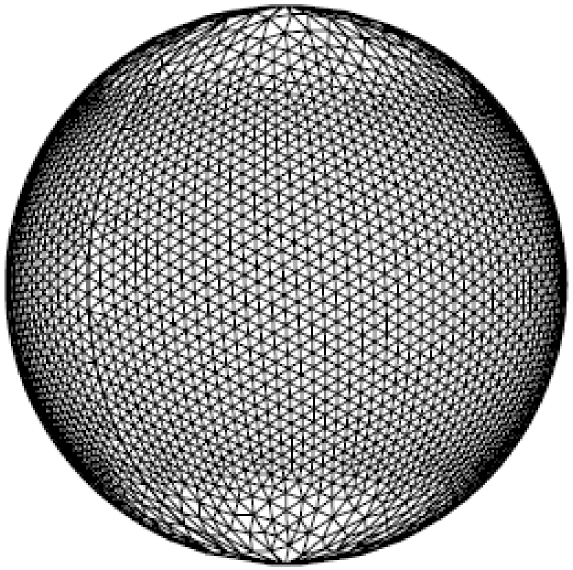
$$\frac{\partial M}{\partial \zeta} = \Pi \,.$$

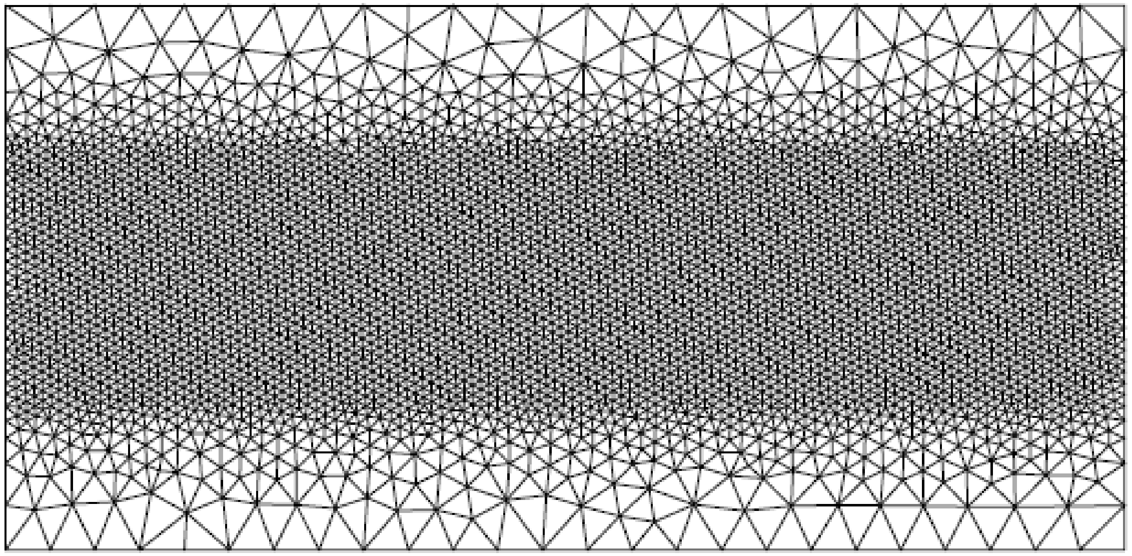
$$\mathcal{D} \equiv \partial p/\partial\zeta$$

$$M \equiv gh + \zeta\Pi$$

Isosteric/isopycninc model: $\zeta = \rho^{-1}$, $\Pi = p$

Isentropic model: $\zeta = \theta$ **,** $\Pi = c_p(p/p_o)^{R_d/c_p}$





(Szmelter & Smolarkiewicz, J. Comput. Phys. 2010)

$Fr=0.5$

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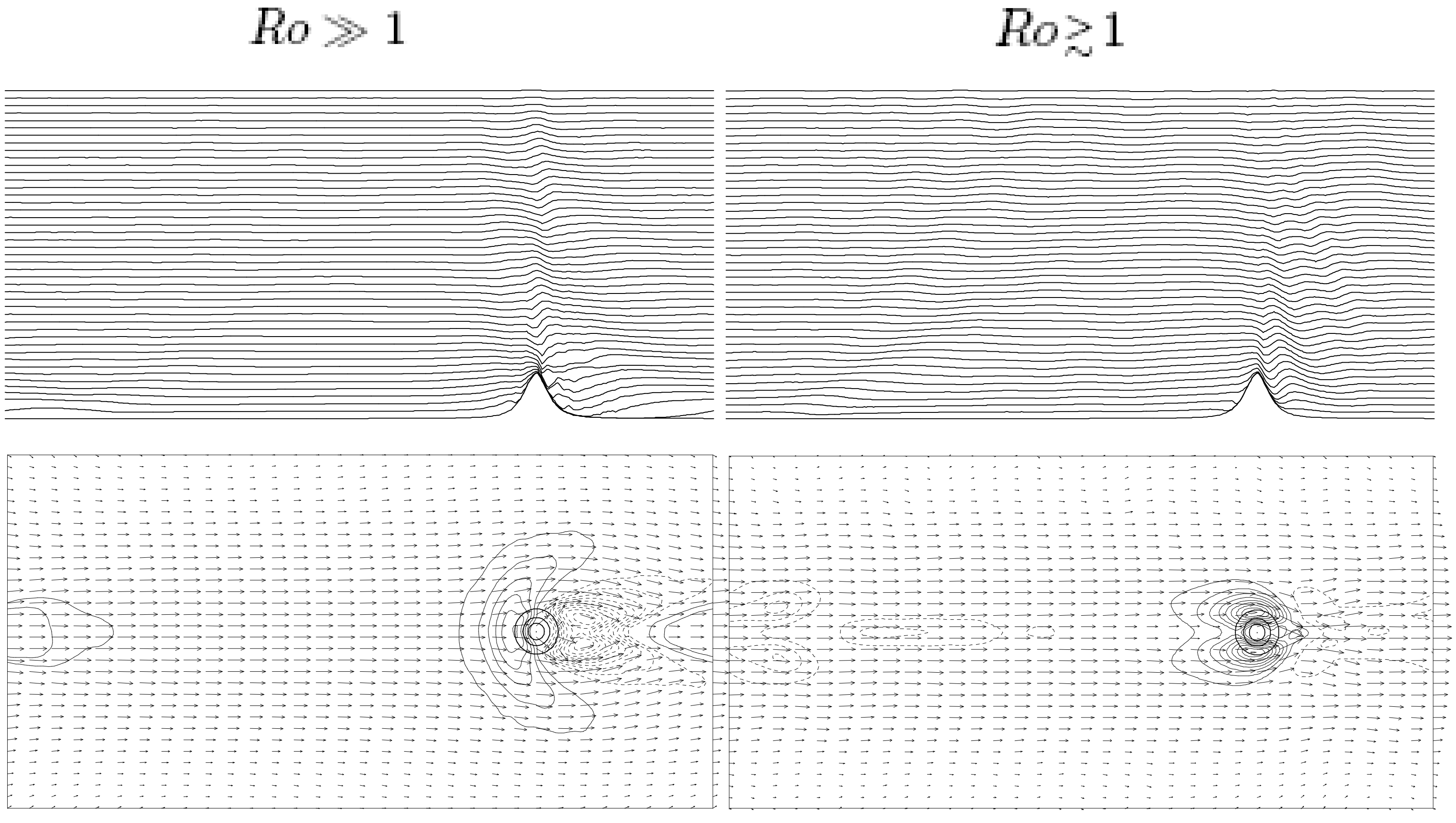

$h_\zeta = 8^{-1}\lambda_o$

Smith, Advances in Geophys 1979; Hunt, Olafsson & Bougeault, QJR 2001

Soundproof generalizations

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Smolarkiewicz & Szmelter, *Acta Geophysica,* 2011

$$\nabla\cdot(\rho^* \mathbf{v}) = 0\,, \quad \frac{D\theta'}{Dt} = -\mathbf{v}\cdot\nabla\theta_e\,, \quad \frac{D\mathbf{v}}{Dt} = -\Theta\nabla\phi' - \mathbf{g}\Upsilon\frac{\theta'}{\bar{\theta}}$$

For [anelastic, pseudo-incompressible]:

$$\rho^* = [\bar{\rho},\ \bar{\rho}\bar{\theta}/\theta_o];\ \Theta = [1,\ \theta/\theta_o];\ \text{and}\ \Upsilon = [1,\ \bar{\theta}/\theta_e]$$

$$\theta' = \theta - \theta_e \qquad \phi' = [(p-p_e)/\bar{\rho},\ c_p(\pi-\pi_e)\theta_o]$$

$$0 = -\nabla\frac{p_e-\bar{p}}{\bar{\rho}} - \mathbf{g}\frac{\theta_e-\bar{\theta}}{\bar{\theta}} \qquad 0 = -c_p\theta_e\nabla(\pi_e-\bar{\pi}) - \mathbf{g}\frac{\theta_e-\bar{\theta}}{\bar{\theta}}$$

Numerics:

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$$\frac{\partial \rho^* \psi}{\partial t} + \nabla \cdot (\rho^* \mathbf{v} \psi) = \rho^* R \;\rightarrow\; \psi_i^{n+1} = \mathcal{A}_i(\tilde{\psi}, \mathbf{v}^{n+1/2}, \rho^*) + 0.5\delta t R_i^{n+1} \equiv \widehat{\psi}_i + 0.5\delta t R_i^{n+1}$$

$$\begin{aligned}
u &= \hat{u} - \delta_h t\, \Theta\, \partial_x \phi' - \delta_h t\, \alpha (u - u_e) \\
w &= \hat{w} - \delta_h t\, \Theta\, \partial_z \phi' + \delta_h t\, \beta \theta' - \delta_h t\, \alpha (w - w_e) \\
\theta' &= \hat{\theta}' - \delta_h t\, w\, \partial_z \theta_e - \delta_h t\, \alpha' \theta' ,
\end{aligned}$$

$$\begin{aligned}
u &= \hat{\hat{u}} - C^{xx} \partial_x \phi' \\
w &= \hat{\hat{w}} - C^{zz} \partial_z \phi'
\end{aligned} \qquad \theta' = \frac{\hat{\theta}' - \delta_h t\, w\, \partial_z \theta_e}{1 + \delta_h t\, \alpha'}$$

$$\hat{\hat{u}} = \frac{\hat{u} + \delta_h t\, \alpha u_e}{1 + \delta_h t\, \alpha}, \quad \hat{\hat{w}} = \frac{\hat{w} + \delta_h t\, \alpha w_e + \delta_h t\, \beta\, \hat{\theta}' (1 + \delta_h t \alpha')^{-1}}{(1 + \delta_h t\, \alpha)\cancel{(1 + \delta_h t\, \alpha')} + (\delta_h t)^2 \beta\, \partial_z \theta_e (1 + \delta_h t \alpha')^{-1}}$$

$$C^{xx} = \frac{\delta_h t\, \breve{\Theta}}{(1 + \delta_h t\, \alpha)}, \quad C^{zz} = \frac{\delta_h t\, \breve{\Theta}}{(1 + \delta_h t\, \alpha)\cancel{(1 + \delta_h t\, \alpha')} + (\delta_h t)^2 \beta\, \partial_z \theta_e (1 + \delta_h t \alpha')^{-1}}$$

$$\frac{1}{\rho^*}\left[\partial_x \rho^* (\hat{\hat{u}} - C^{xx} \partial_x \phi') + \partial_z \rho^* (\hat{\hat{w}} - C^{zz} \partial_z \phi')\right] \equiv -(\mathcal{L}\phi' - \mathcal{R}) = 0$$

Non-Boussinesq amplification and breaking of deep stratospheric gravity wave

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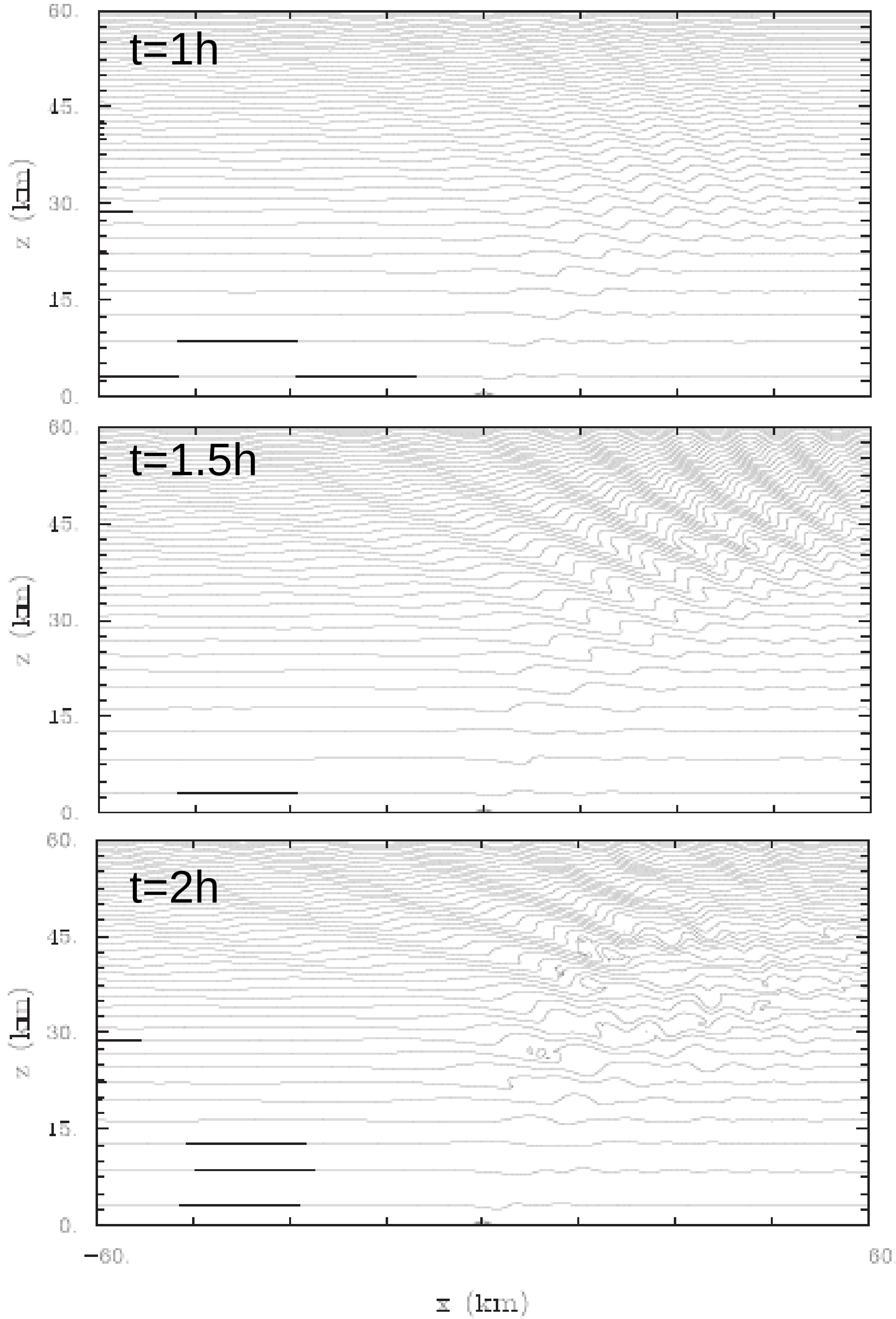
isothermal reference profiles ; $H_\theta = 3.5\, H_\rho$

$NL/U_o \approx 1$, $Fr \approx 1.6$;
$\lambda_o = 2\pi$ km $\quad H_\rho \Rightarrow$
$A(H/2)=10h_o = \lambda_o$

EULAG "reference" solution using terrain-following coordinates ➔



Prusa et al., *JAS* 1996;
Smolarkiewicz & Margolin, *Atmos. Ocean*,1997;
Klein, *Ann. Rev. Fluid Dyn.*, 2010

unstructured-mesh “EULAG” solution, using mesh (left) mimicking terrain-following coordinates, and (right) a fully unstructured mesh

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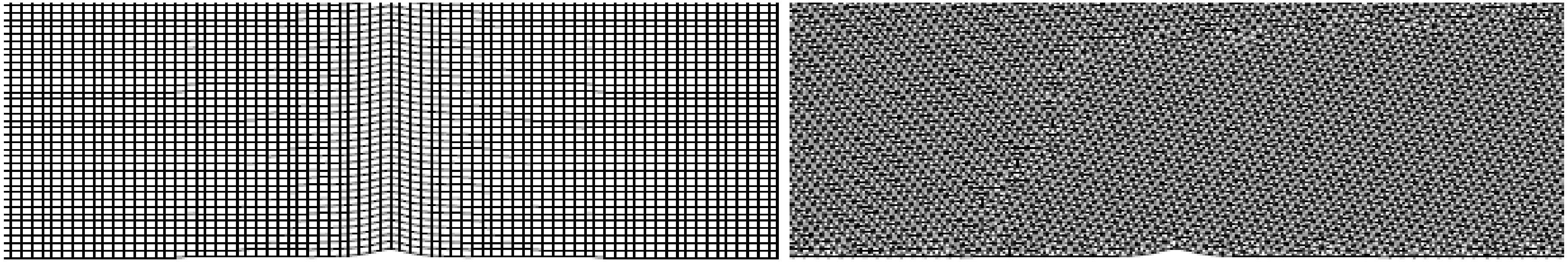
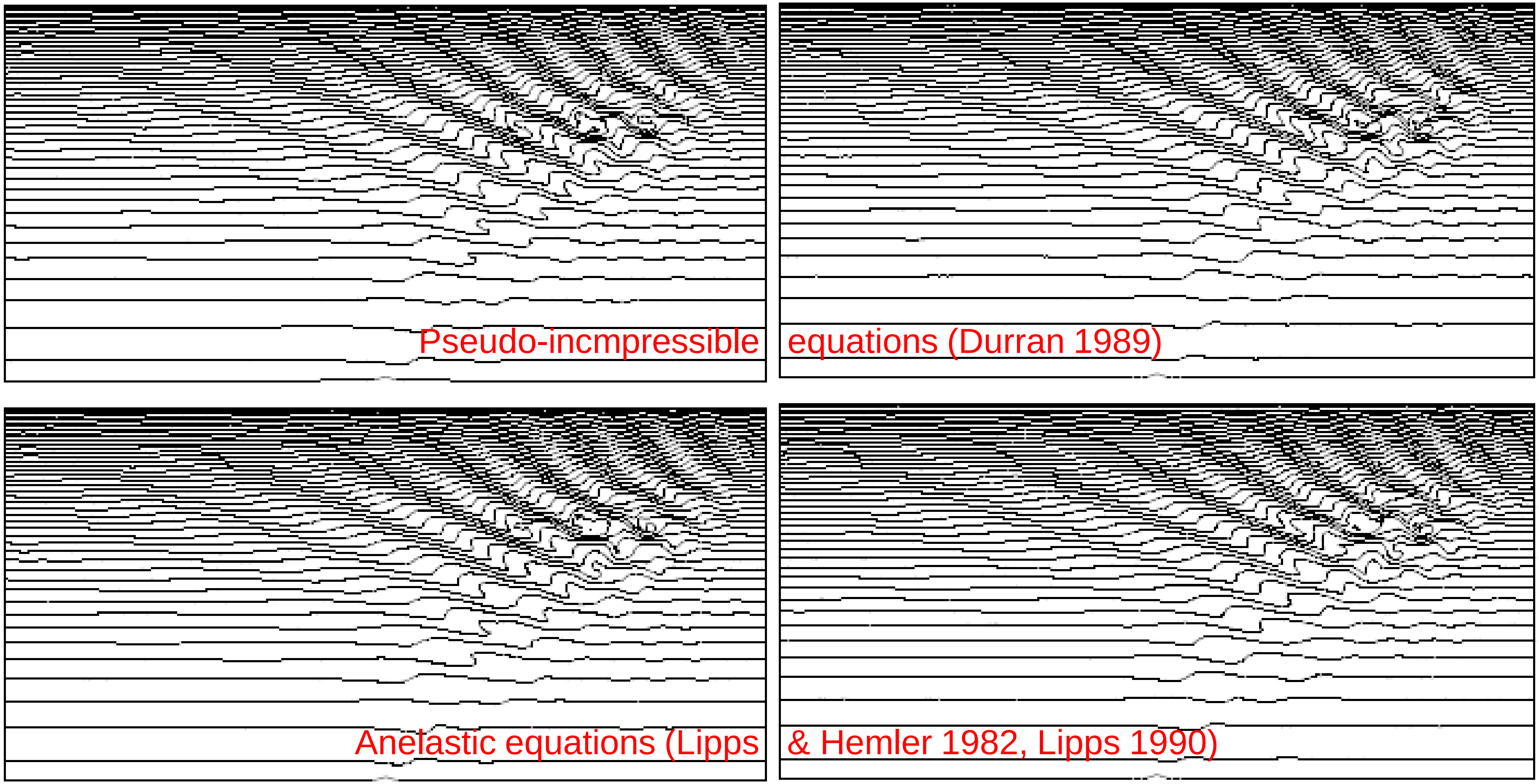


t=1.5h



H_θ H_ρ (RE: Achatz et al., *JFM,* 2010)

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t=0.75h

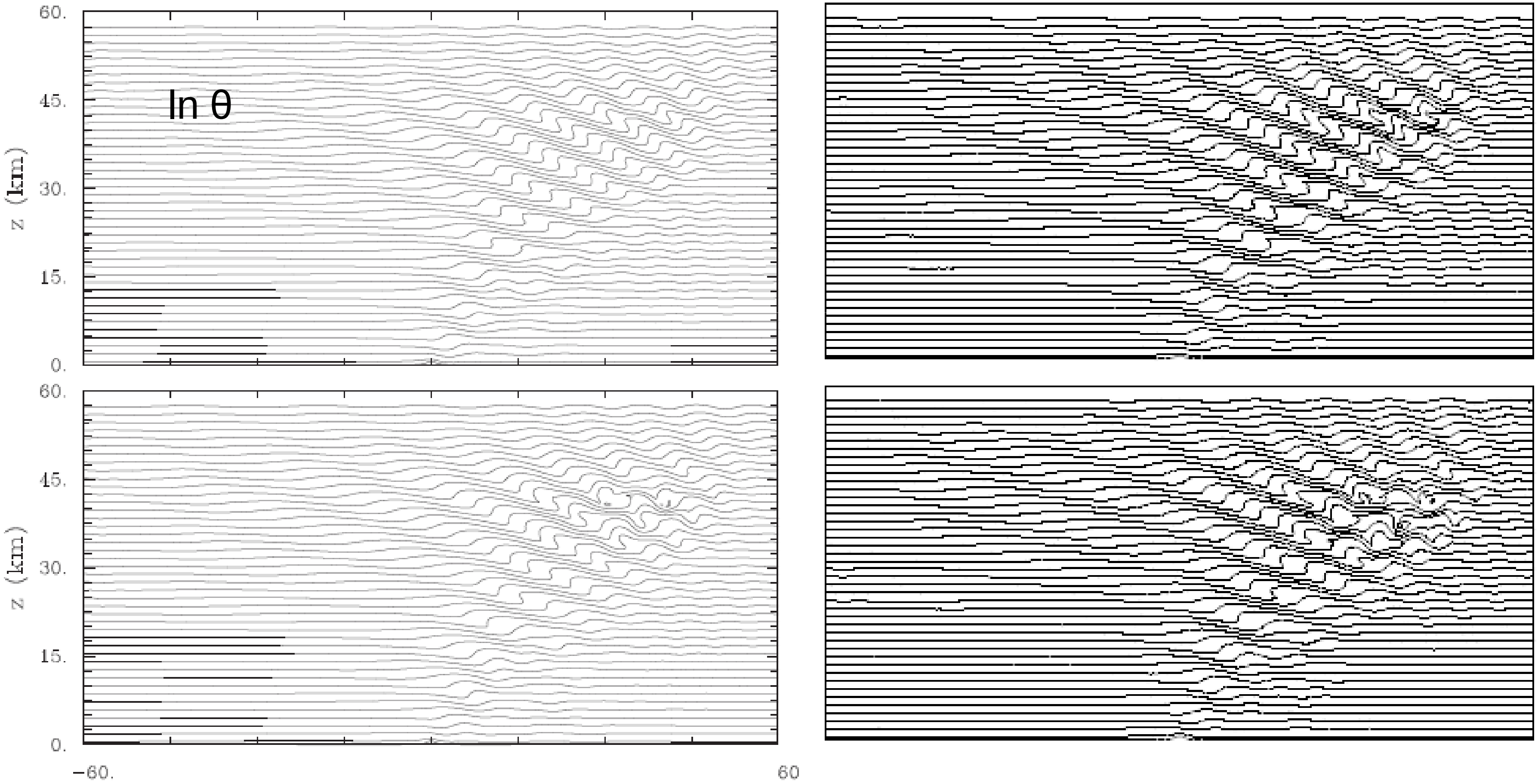

Table 1: Normalized vorticity: maximum, minimum, mean and standard deviation.

eqs.	*numerics*	$\max(\omega)$	$\min(\omega)$	$\overline{\omega}$	$\overline{(\omega-\overline{\omega})^2}^{1/2}$
PSI	CV/grid	0.17	-0.21	$1.6 \cdot 10^{-4}$	$3.3 \cdot 10^{-2}$
ANL	CV/grid	0.27	-0.41	$6.4 \cdot 10^{-5}$	$3.5 \cdot 10^{-2}$
PSI	CV/mesh	0.28	-0.24	$2.0 \cdot 10^{-4}$	$3.7 \cdot 10^{-2}$
ANL	CV/mesh	0.24	-0.36	$9.5 \cdot 10^{-5}$	$3.6 \cdot 10^{-2}$
PSI	SL/grid	0.28	-0.30	$2.1 \cdot 10^{-4}$	$3.1 \cdot 10^{-2}$
ANL	SL/grid	0.18	-0.24	$7.2 \cdot 10^{-5}$	$3.0 \cdot 10^{-2}$

Conclusions:

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Unstructured-mesh discretization can sustain the accuracy of structured-grid discretization while providing full flexibility in spatial resolution.

Soundproof models are effective for a broad range of atmospheric flows and have numerical advantages over fully compressible models.