Convergence Properties of

Higher Order Advection Schemes

In the COSMO Model

CONSORTIUM FOR SMALL SCALE MODELING

- first results and model developments -

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Outline:

1. Motivation
2. Higher order (explicit) spatial schemes
3. Idealised test case results
4. Model developments
5. Summary and conclusions

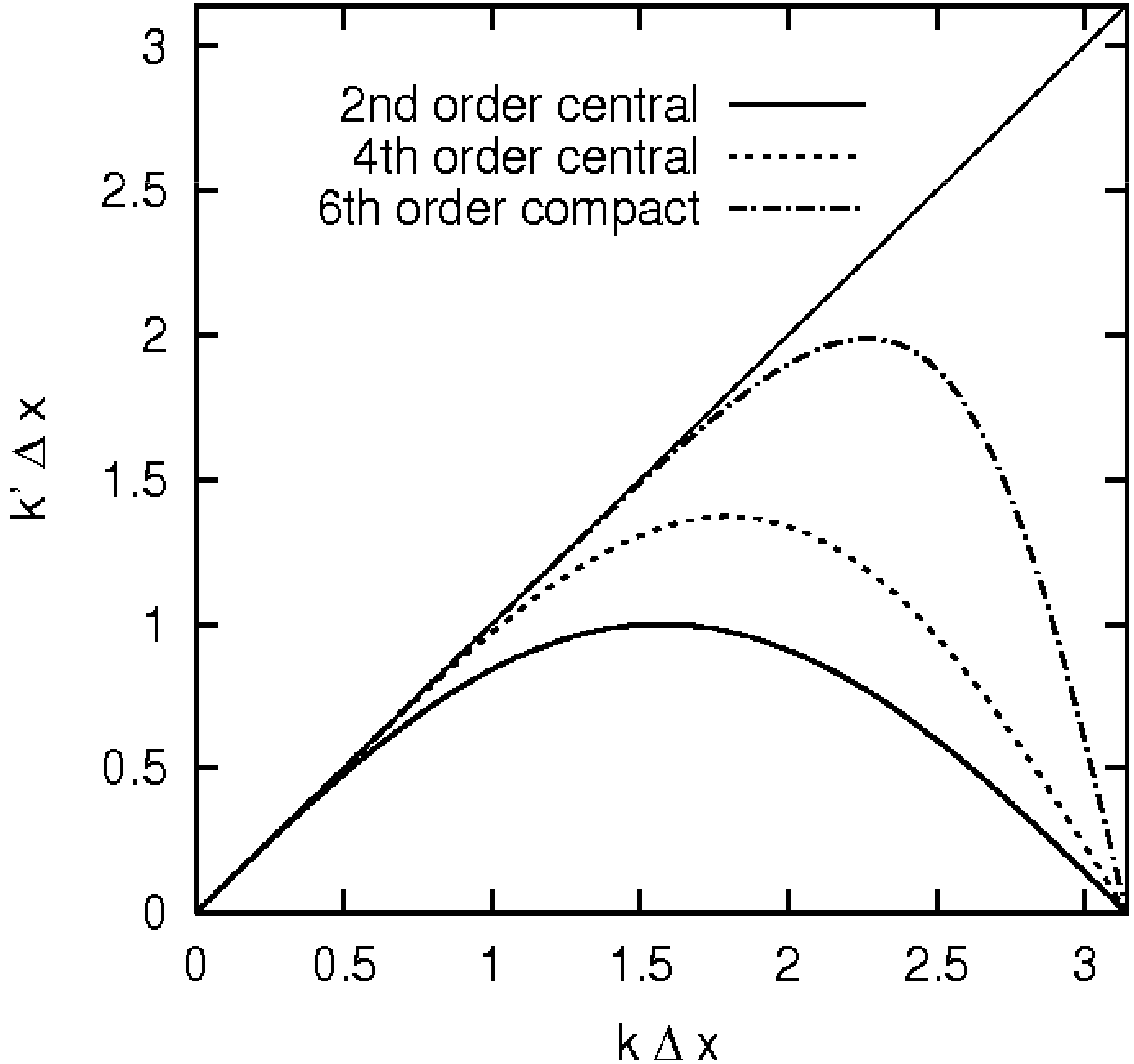
1. Motivation:

Development of

scale selective (i.e. with no phase and amplitude error at resolved scales), → **High order spatial schemes**

and conservative (at least mass) → **non-dissipative dynamics, no numerical diffusion, skew-symmetric schemes**

1. Motivation



Modified Wavenumber

and

Phase Error

Fig. H.-J.Kaltenbach

2. Higher order explicite spatial schemes

1. **Construction of centered differences higher order advection schemes on a staggered grid**
2. **Wicker-Skamarock (2002)**
3. **COSMO, Baldauf (2008)**
4. **Incompressible, Morinishi (1998)**

2.1 Centered differences higher order advection

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Centered differences:

$$\delta_{n\lambda}\psi = \frac{\psi(\lambda + n\,h_1/2, \phi, z) - \psi(\lambda - n\,h_1/2, \phi, z)}{n\,h_1} = \frac{\psi_{i+n/2,j,k} - \psi_{i-n/2,j,k}}{n\,h_1}$$

1stD

$$\delta_{n\lambda}\psi = \partial_\lambda \psi + \sum_{i=1}^{\infty} \frac{n^{2i}\,\partial_\lambda^{2i+1}\psi}{(2i+1)!\;2^{2i}}\,h^{2i}$$

Taylor exp. of RHS of 1stD

2nd order interpolation

$$\overline{\psi}^{n\lambda} = \frac{\psi(\lambda + nh_1/2, \phi, z) + \psi(\lambda - nh_1/2, \phi, z)}{2} = \frac{\psi_{i+n/2,j,k} + \psi_{i-n/2,j,k}}{2}$$

INT

$$\overline{\psi}^{n\lambda} = \psi + \sum_{i=1}^{\infty} \frac{n^{2i}\,\partial_\lambda^{2i}\psi}{2i!\;2^{2i}}\,h^{2i}$$

Taylor exp. of RHS of INT

Higher order advection on a staggered grid

$$\overline{\psi}^{On,\lambda} = \sum_n a_n \overline{\psi}^{n\lambda}$$

$$\delta_\lambda^{On}\psi = \sum_n a_n \delta_{n\lambda}\psi \quad \text{with} \quad n\varepsilon\{n_1, \cdots, n_{m/2}\}$$

$$(\mathbf{v}\cdot\nabla\mathbf{v})_{\mathrm{i}}^{\mathrm{On}} = \mathrm{u}\delta_\lambda^{\mathrm{On}}\mathrm{u} + \overline{\overline{\mathrm{v}}^{\mathrm{On},\phi}}^{\mathrm{On},\lambda}\delta_\phi^{\mathrm{On}}\mathrm{u} + \overline{\overline{\mathrm{w}}^{\mathrm{On,z}}}^{\mathrm{On},\lambda}\delta_{\mathrm{z}}^{\mathrm{On}}\mathrm{u}$$

4th

$$(\mathbf{v}\cdot\nabla\mathbf{v})_{\mathrm{i}}^{\mathrm{Sm}} = \overline{\overline{\mathrm{u}}^{\mathrm{Sm,i}}\delta_{\mathrm{i}}^{\mathrm{Sm}}\mathrm{u}}^{\mathrm{Sm,i}} + \overline{\overline{\mathrm{v}}^{\mathrm{Sm,j}}\delta_{\mathrm{i}}^{\mathrm{Sm}}\mathrm{u}}^{\mathrm{Sm,i}} + \overline{\overline{\mathrm{w}}^{\mathrm{Sm,k}}\delta_{\mathrm{k}}^{\mathrm{Sm}}\mathrm{u}}^{\mathrm{Sm,i}}$$

Morinishi et al. 1998:
Sm conserves
the kinetic energy!!!

2.2 Flux schemes of Wicker/Skamarock

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Spatial scheme order 4:

$$\nabla(\mathbf{v}\mathrm{q})_{\mathrm{i}}^{\mathrm{WS}} = \frac{1}{12\Delta x}\Big\{u_{i+1/2}\Big[7(q_{i+1}+q_i)-(q_{i+2}+q_{i-1})\Big] - u_{i-1/2}\Big[7(q_i+q_{i-1})-(q_{i+1}+q_{i-2})\Big]\Big\}$$

General form:

$$\nabla(\mathbf{v}\mathrm{q})_{\mathrm{i}}^{\mathrm{WS}} = \delta_i^{S1}(uq) = \delta_i^{S1}u(b_1\overline{q}^{S1,i} + b_3\overline{q}^{S3,i})$$

$$\text{with} \quad b_1 = 7/6 \quad \text{and} \quad b_3 = -1/6$$

1stD of flux F=u q combined with
a combination of
S1 and S3 interpolation of q

Taylor expansion of the scheme:

$$\nabla(\mathbf{v}\mathrm{q})_{\mathrm{i}}^{\mathrm{WS}} = \nabla(\mathbf{v}\mathrm{q})_{\mathrm{i}} + \frac{1}{24}\Big[\partial_i^3(u_i q_i) - (u_{i+1/2}\partial_i^2 q_{i+1/2} - u_{i-1/2}\partial_i^2 q_{i-1/2})\Big](\Delta x)^2 + O((\Delta x)^4)$$

Taylor exp. of WS scheme: **4th order** for special cases only, e.g. 1D advection with constant u

2.3 COSMO schemes for the advection operator

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Spatial schemes order 1 … 6 (Baldauf (2008)

$$f_j^{(1)}(q) := -u \frac{q_j - q_{j-1}}{\Delta x}$$ up 1st order

$$f_j^{(2)}(q) := -u \frac{q_{j+1} - q_{j-1}}{2\,\Delta x}$$ cd 2nd order

$$f_j^{(3)}(q) := -u \frac{2q_{j+1} + 3q_j - 6q_{j-1} + q_{j-2}}{6\,\Delta x}$$ up 3rd order

$$f_j^{(4)}(q) := -u \frac{-(q_{j+2} - q_{j-2}) + 8(q_{j+1} - q_{j-1})}{12\,\Delta x}$$ cd 4th order

$$f_j^{(5)}(q) := -u \frac{-3q_{j+2} + 30q_{j+1} + 20q_j - 60q_{j-1} + 15q_{j-2} - 2q_{j-3}}{60\,\Delta x}$$ up 5th order

$$f_j^{(6)}(q) := -u \frac{(q_{j+3} - q_{j-3}) - 9(q_{j+2} - q_{j-2}) + 45(q_{j+1} - q_{j-1})}{60\,\Delta x}$$ cd 6th order

General form of the 4th order scheme:

$$(\mathbf{v} \cdot \nabla \mathbf{v})_{\mathrm{i}}^{\mathrm{O4}} = u\delta_\lambda^{On} u + \overline{\overline{v}^{O2,\phi}}^{O2,\lambda} \delta_\phi^{On} u + \overline{\overline{w}^{O2,z}}^{O2,\lambda} \delta_z^{On} u$$

$$\text{with} \quad a_2 = 4/3 \quad \text{and} \quad a_4 = -1/3$$

COSMO advection schemes
are high order accurate
for special cases only,
e.g. 1D advection of u

3. Results of idealised tests

1. **Small disturbance growth in a channel flow: Amplitude and Phase error of 2nd to 6th order schemes**
2. **2D mountain flow in the COSMO model: Convergence of 2nd to 6th order COSMO advection schemes**

3.Results of idealised tests

1. **Small disturbance growth in a channel flow: Amplitude and Phase error of 2nd to 6th order schemes**

scheme	form of	quadratic and viscous terms		$\partial p/\partial q$ and $\partial u_q/\partial q$
		$\partial/\partial x, \partial/\partial z$	$\partial/\partial y$	for $q = x$ and $q = z$
xz2y2p2e	divergence	2nd expl.	2nd expl.	2nd expl.
xz4y2p4e	divergence	4th expl.	2nd expl.	4th expl.
xz6y2p2e	skew-sym.	6th impl.	2nd expl.	2nd expl.
xz6y2p4e	skew-sym.	6th impl.	2nd expl.	4th expl.
xz6y2p6e	skew-sym.	6th impl.	2nd expl.	6th expl.
xz6y4p6e	skew-sym.	6th impl.	4th expl.	6th expl.

Results of SDG: **Energy growth for n_x=8, n_y=256**

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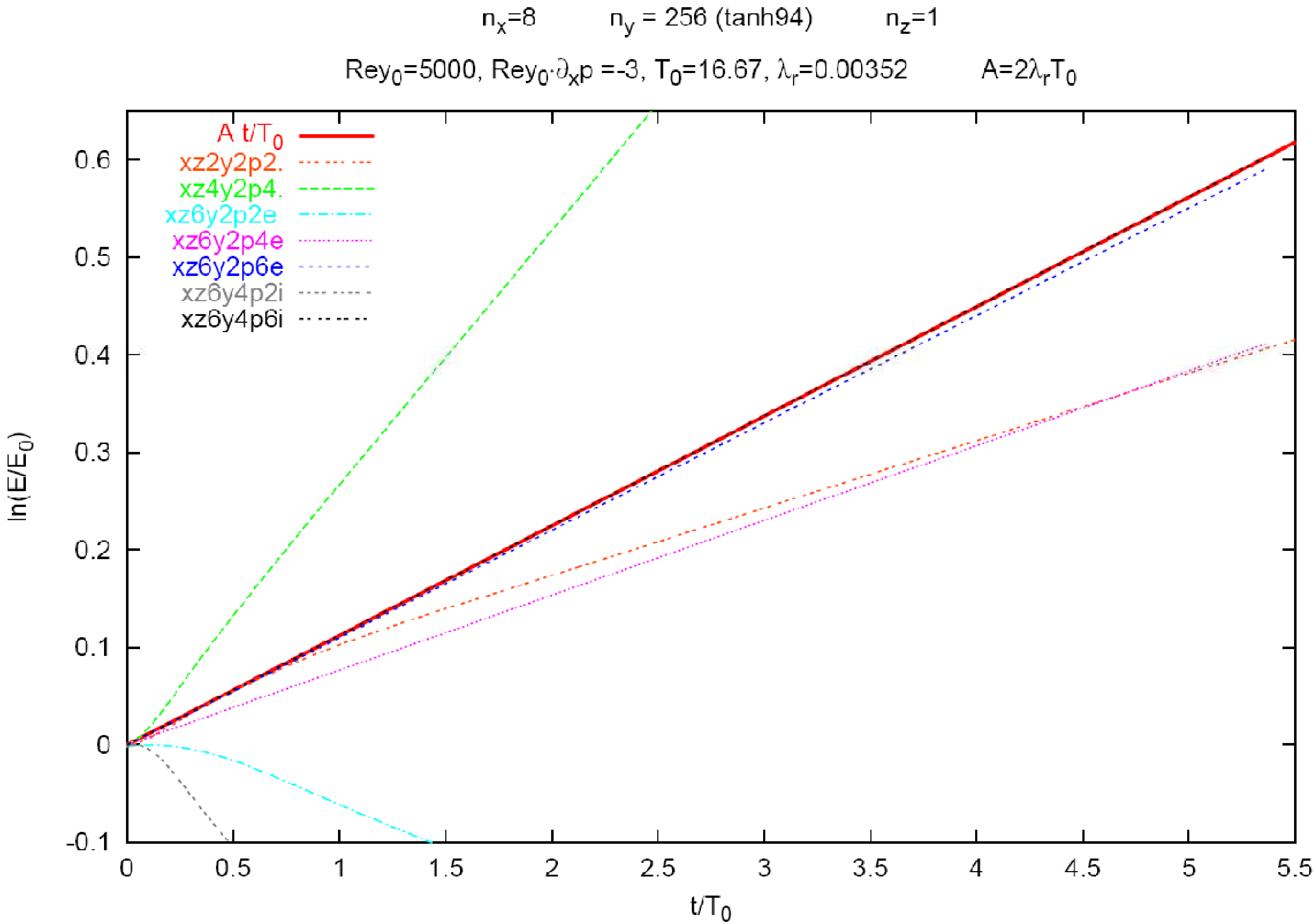

3.2 Results of idealised tests

2. 2D mountain flow in the COSMO model: Convergence of 2nd to 6th order COSMO advection schemes

3.2 Idealized Test Case: 2D Mountain Flow, System of Equations

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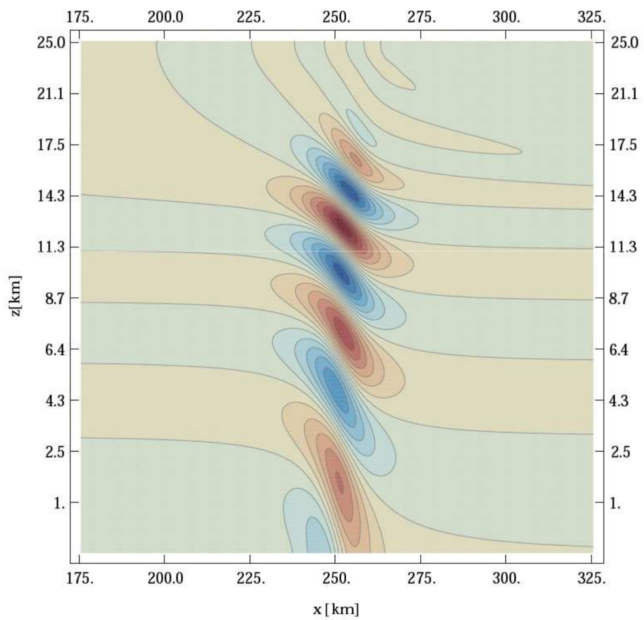
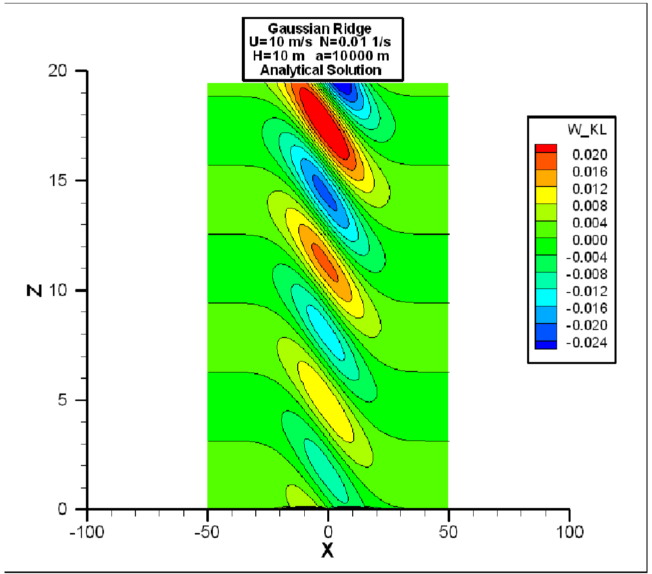
Numerical Solution (left)
Dx=500m

Analytical solution (right)
(Klemp-Lilly (1978) *JAS*)





2D mountain flow
Linear hydrostatic regime:
prerequisites

- **no friction**
- **adiabatic processes**
- **Ideal gas law**
- **no earth curvature**
- **no coriolis force**

$$\frac{\partial u}{\partial t} + \mathbf{v}.\nabla u = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \mathbf{v}.\nabla v = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\delta_1 \frac{\partial w}{\partial t} + \delta_1 \mathbf{v}.\nabla w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

$$\delta_2 \frac{\partial p}{\partial t} + \delta_3 \mathbf{v}.\nabla \rho + \rho \nabla.\mathbf{v} = 0$$

$$\frac{dp}{dt} = c^2 \frac{d\rho}{dt} \ , \ c^2 := \frac{c_p}{c_v}\frac{p}{\rho} \ , p = \rho R T$$

$\delta_1 = 0/1$: hydrostatic / non-hydrostatic approximation $\delta_2 = 0/1$: incompressible / compressible $\delta_3 = 0/1$: shallow / deep atmosphere

3.2 2DM idealized test case: basic configuration

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stationary mountain flow, hydrostatic linear case (Baldauf, 2009)

Initial conditions: $w_0=0$, $u_0=10m/s$

Typical quantities: $u_0=10m/s$, $t_0= a/u_0=1000s$

$a=10km$ (mountain half width) $h= 10m$

$N=0.01$

Domain: $Lx=50\ a$, $Lz = 1.95\ a\ (2.5a)$, $t = 360\ t_0$

horizontal resolution: dx^*: 0.0125 to 0.4

CFL = 0.05,

$ke=195\ (500)$, $\Delta z^* \approx 0.01\ (0.005)$ (stretched)

4. 2DM idealized test case: Results

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Calculaiton of error norms; L0, L1, L2

$$\mathcal{L}_0 = \max_{i=I1,k=K1}^{I2,K2} \parallel F_{\Delta x}(\lambda_i, z_j, t) - F_{Ref} \parallel$$

$$\mathcal{L}_1 = \frac{1}{I \cdot K} \sum_{i=I1,k=K1}^{I2,K2} \parallel F_{\Delta x}(\lambda_i, z_j, t) - F_{Ref} \parallel$$

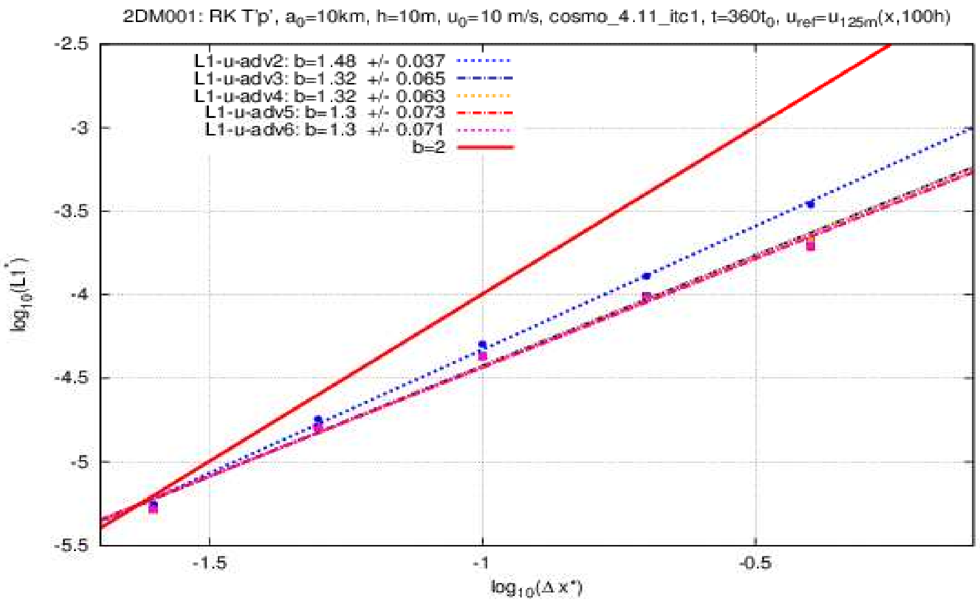
$$\mathcal{L}_2 = \sqrt{\frac{1}{I \cdot K} \sum_{i=I1,k=K1}^{I2,K2} (F_{\Delta x}(\lambda_i, z_j, t) - F_{Ref})^2}$$

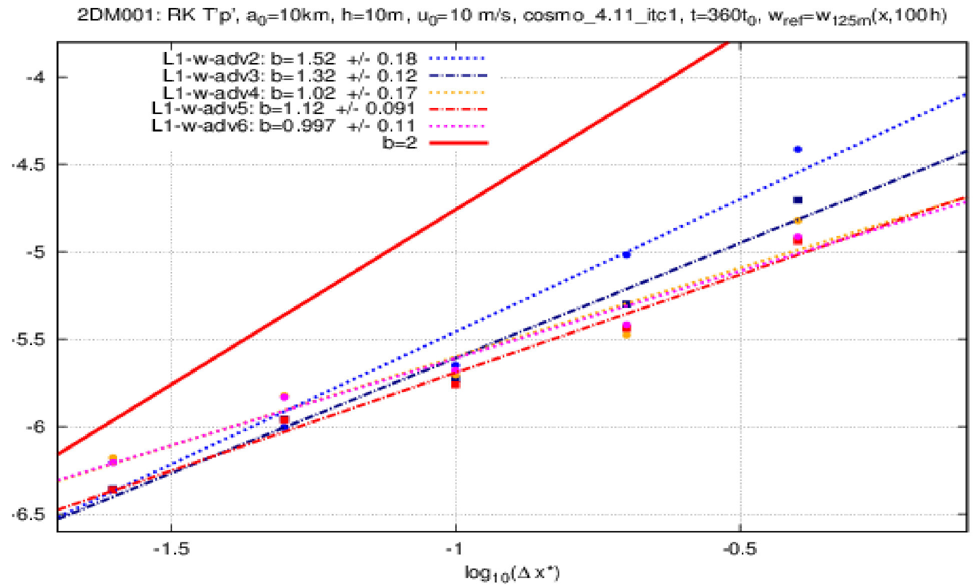
$$\text{with} \quad I = I2 - I1 + 1, \quad K = K2 - K1 + 1$$

L_0 = maximum difference
L_1 = mean absolute difference
L_2 = root mean square difference

2D idealized test cases: Configuration 1

L1 (u), Ref: 125m solution L1(w)





- Significant difference betwwen 2nd other and the higher order schemes
-Same slopes for all norms of u,
-inconsistent with theory!

- Slopes inconsistent for different norms of w (intersection of lines)
- Slopes inconsistent with those of u
- Different convergence of different schemes

3.2 2DM idealized test case: acuracy requirements

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See poster Ogaja/Will for more details

The assumptions of the test case and other model error sources have to be one order of magnitude smaller than the discretisation error investigated. Otherwise the convergence properties are significantly affected. This means:

1. Assumptions of the test case:

1.1. w=0 : solution of the discretised equation for the vertical pressure profile

1.2 Stationarity : t=360 t*=100h

2. Main Error sources

2.1 Time discretisation : CFL = 0.05

2.2 Vertical discretisation : ke=500 , $\Delta z^* \approx 0.005$ (stretched, 10m to 100m)

2.3 LB relaxation : exp(-12 x/x0)

2.4 UB Rayleigh damping : $\sin^2$(Pi x/x0) with tau=c*dt=400, x0=12km

3.2 2D idealized test cases: w(100h)-w(99h), dx=125m, ke=500

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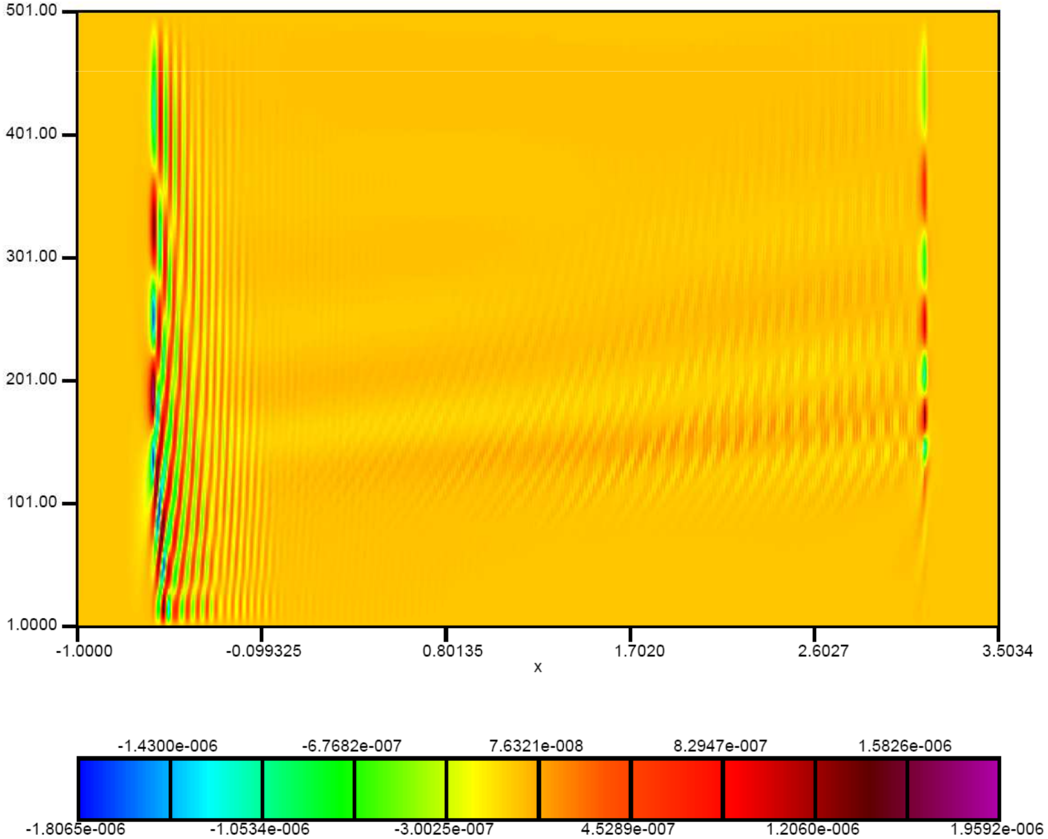
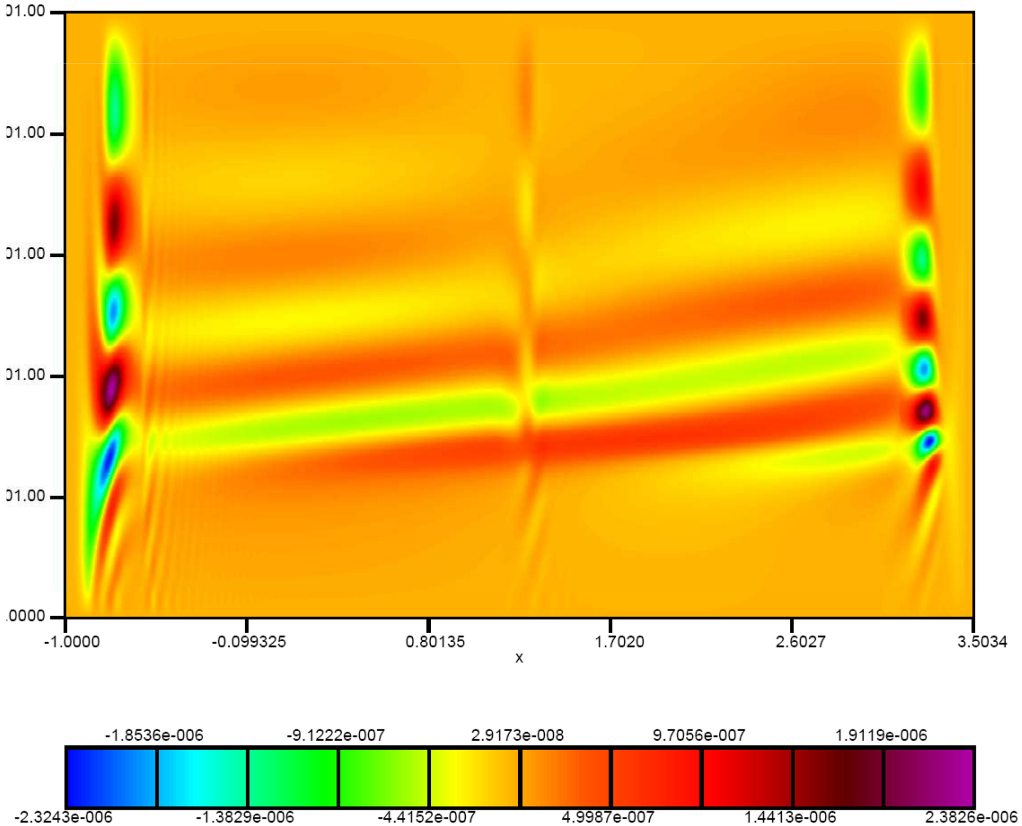
LBC: exp(-6 x/x0)



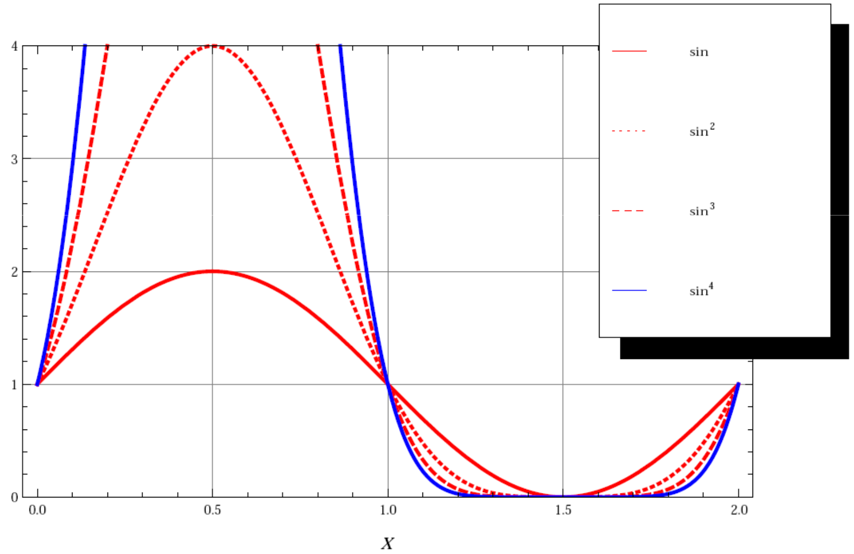
h=10m, $\Delta w=10^{-6}$

LBC: exp(-10 x/x0), RD, small tau



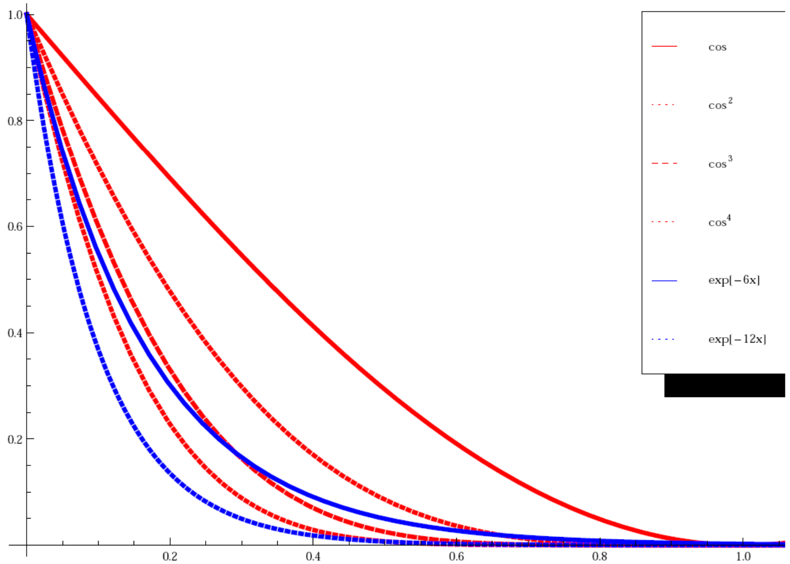
h=1m, $\Delta w=10^{-6}$

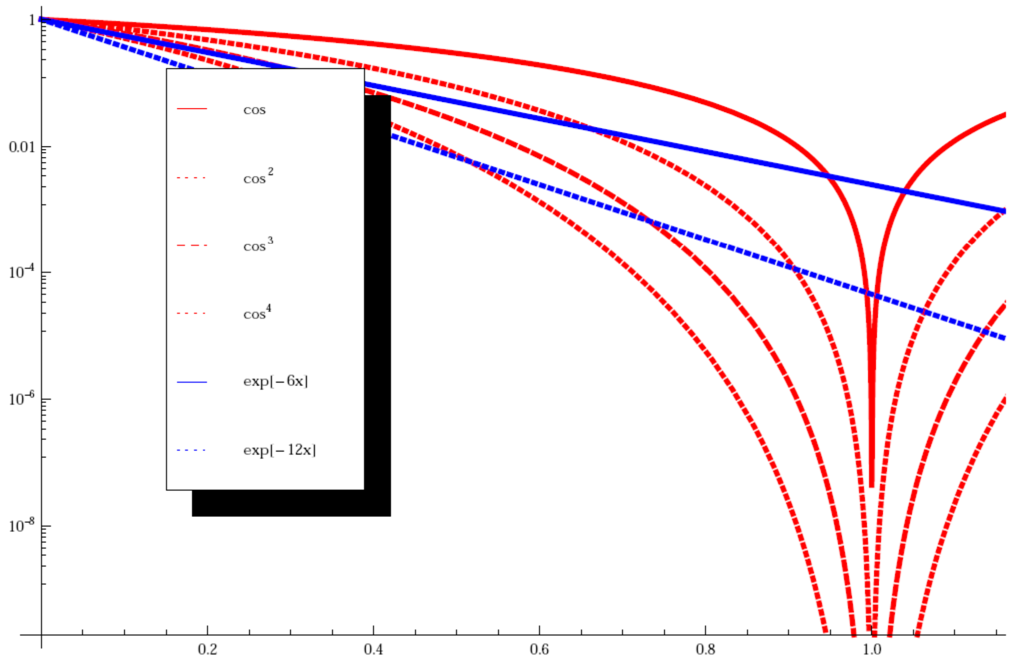
3.2 2DM idealized test cases: New Developments



New Damping function:

$$Fsin(x)=(1+\sin[0.5\ Pi\ x+Pi])^n\ ;\ x\ \epsilon[0,1]$$



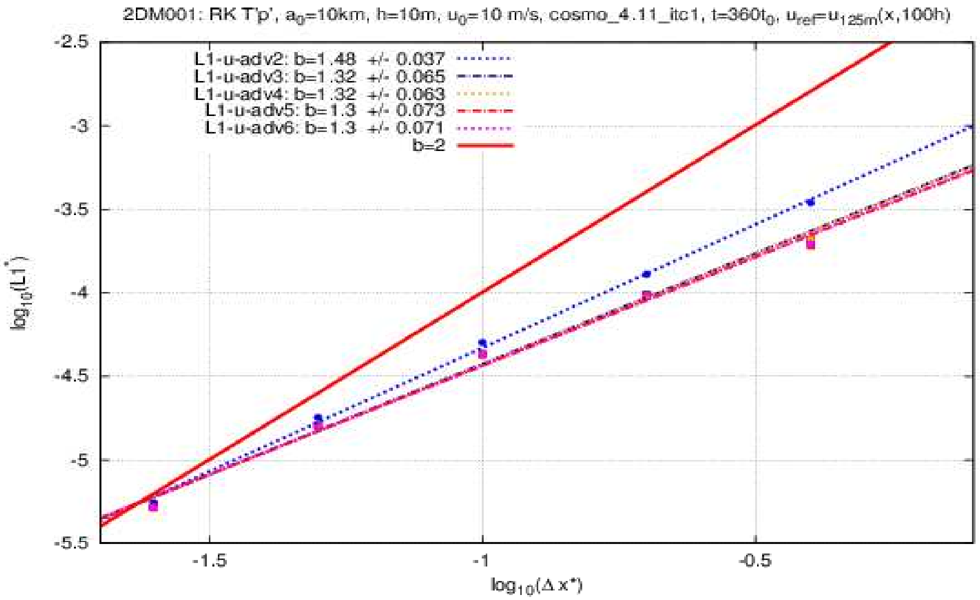


3.2 2DM idealized test cases: New Developments: HO interpolation

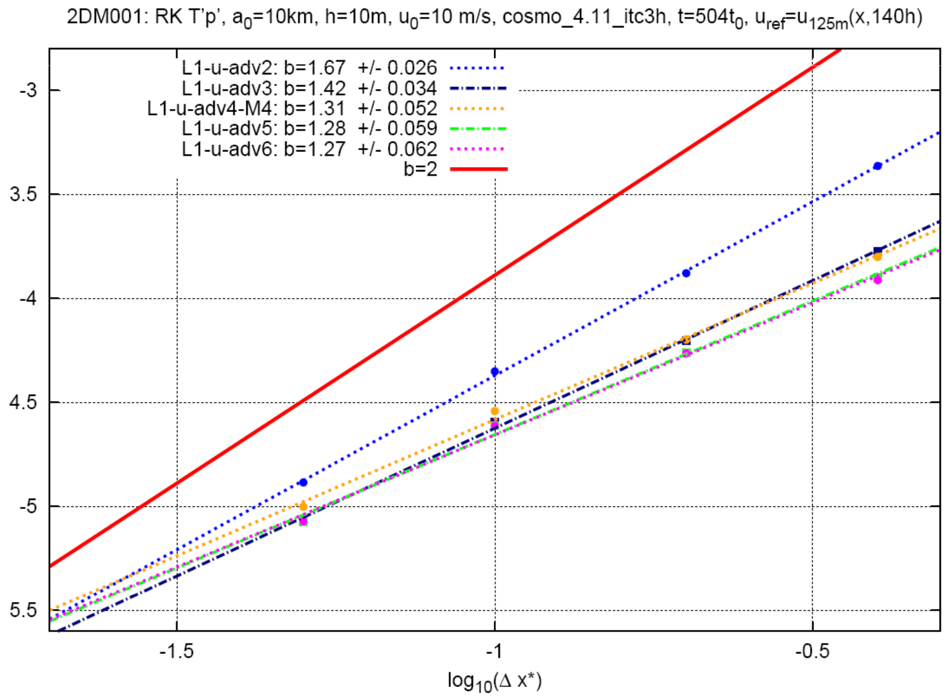
Scheme	Order of convergence for different terms and spatial directions		
	Horizontal		
	Derivative / Interpolation	Pressure term	Vertical derivative / Interp.
adv6	6 central / 2 central	2 central	2 central
adv6-M6	6 central / 6 central	2 central	2 central
adv5	5 upwind / 2 central	2 central	2 central
Adv4	4 central / 2 central	2 central	2 central
Adv4-M4	4 central / 4 central	2 central	2 central
Adv3	3 upwind / 2 central	2 central	2 central
Adv2	2 central / 2 central	2 central	2 central

2D idealized test cases: Configuration ½, L1(u)

Config 1, Ref: dx=125m



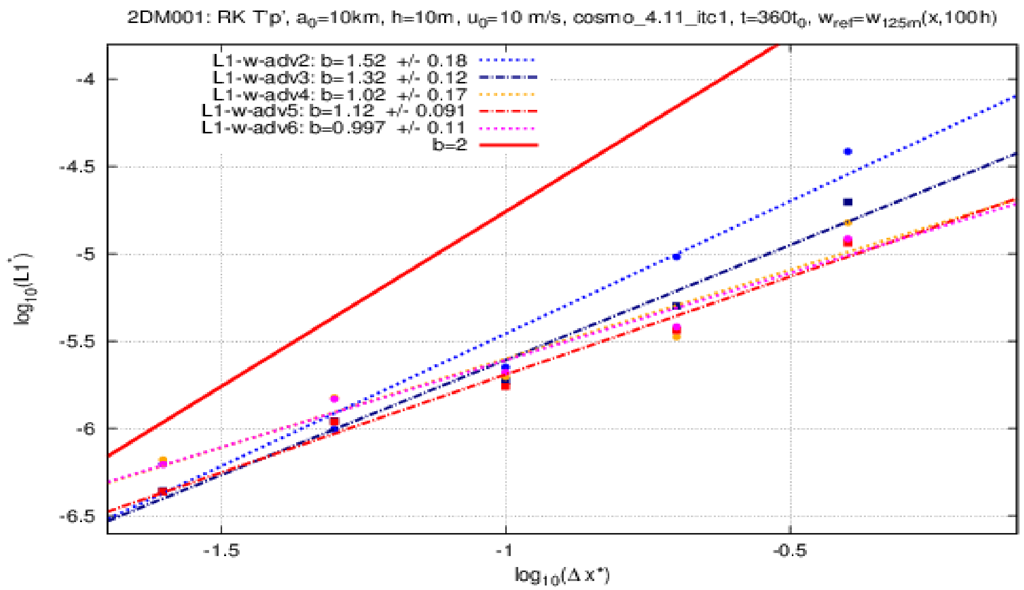
Config2, Ref: dx=250m



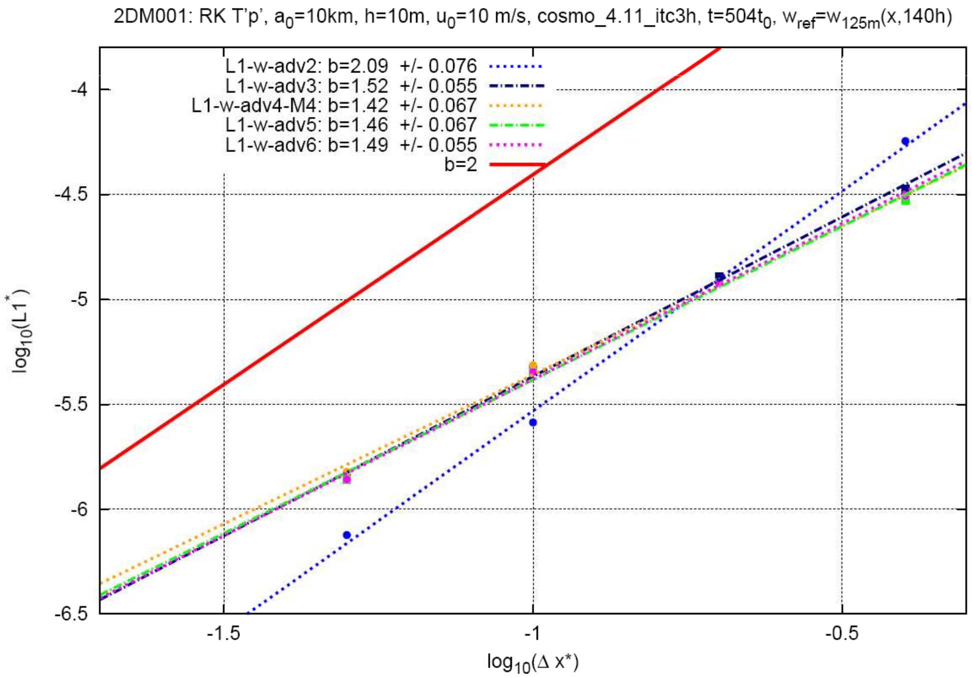
- Higher accuracy
- increased convergence for 2nd order
- no improvement due to 4th order interpolation

2D idealized test cases: Configuration ½, L1(w)

Config 1, Ref: dx=125m



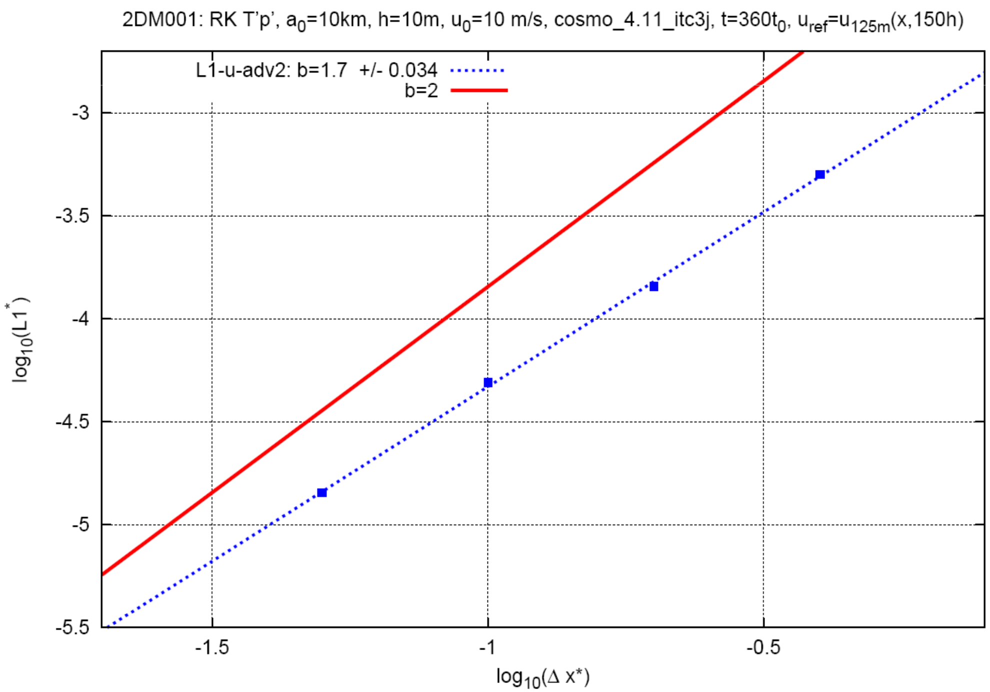
Config2, Ref: dx=250m



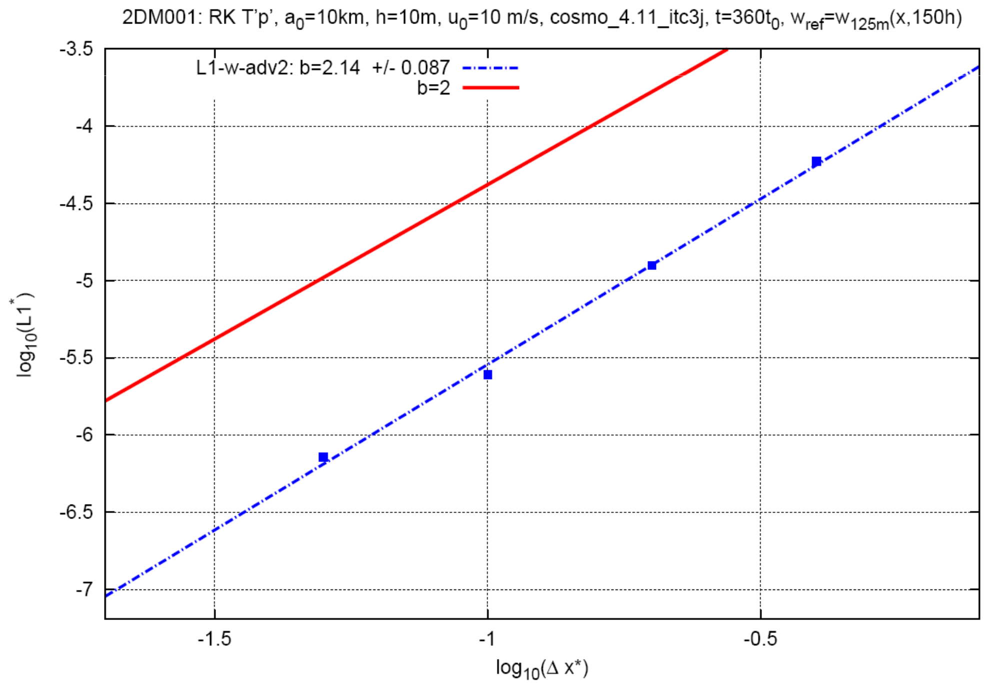
- Higher accuracy
- 2nd order scheme consistent with theory
- consistent results for all HO schemes

2D idealized test cases: RD: $(1+\cos[\mathrm{Pi}/2(x/x0+1)])^4$

u



w



- 2nd order scheme exhibits the same results (without tuning of the coefficient).

Summary

→ the investigation of the convergence properties of numerical schemes is a critical test for all parts of the model involved

→ the COSMO model (and probably other NWP&Climate models too) do not exhibit 2nd order convergence for the higher order advection schemes.

→ The possible reason for is the mixing of the orders of accuracy in the advection and the pressure term

→the realisation of higher order schemes and conservation properties requires the realisation of high order and conservation for all parts of the equation system involved

Thank you for your attention

Discretisation methods: Exlicite versus compact schemes

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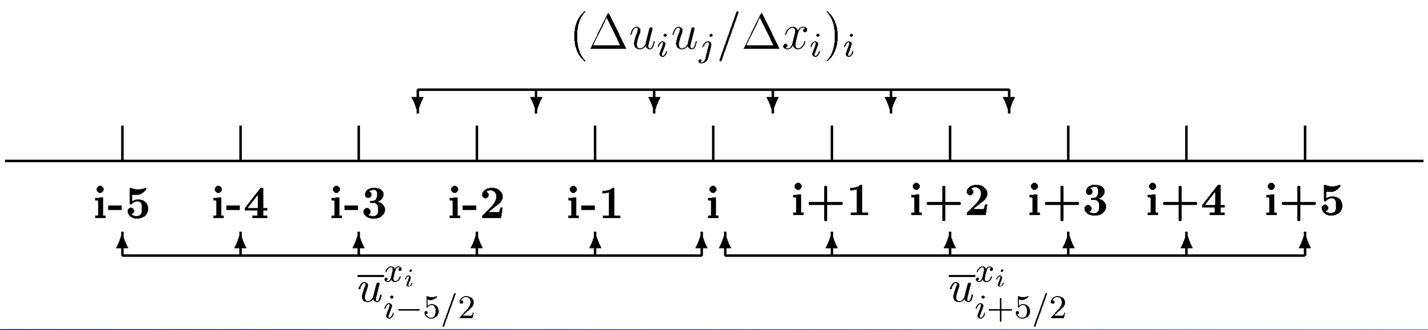
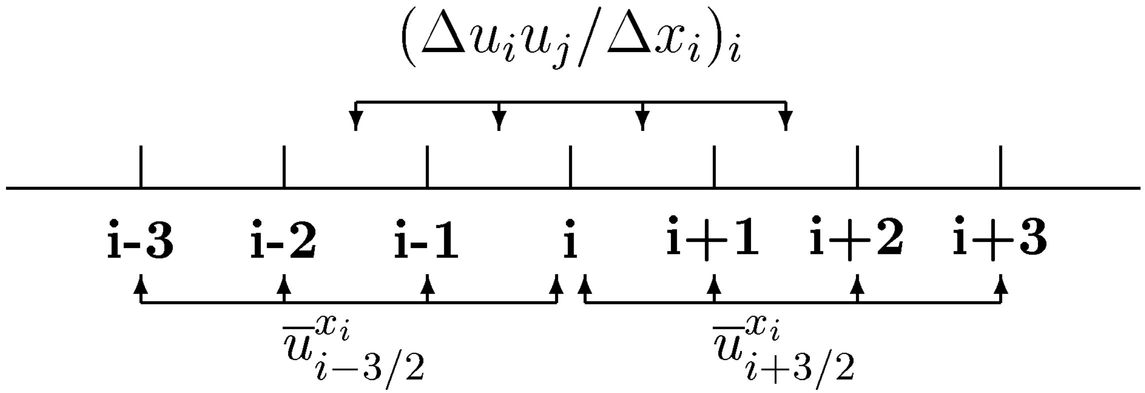
6th order explicite (Mirinishi et al. 1998):



6th order compact (Moin, Kaltenbach):



$0.5(\partial(u_i\, u_j)/\partial\, x_j + u_j\, \partial u_i/\partial x_j)$

skew-symmetric form

$$
\begin{aligned}
\alpha \frac{\Delta u}{\Delta x}\Big|_{i-1} + \frac{\Delta u}{\Delta x}\Big|_{i} + \alpha \frac{\Delta u}{\Delta x}\Big|_{i+1} &= \beta(u_{i+1/2} - u_{i-1/2}) + \gamma(u_{i+3/2} - u_{i-3/2}) \\
\kappa\, \overline{u}|_{i-1} + \overline{u}|_{i} + \kappa\, \overline{u}|_{i+1} &= \mu(u_{i+1/2} + u_{i-1/2}) + \nu(u_{i+3/2} + u_{i-3/2}) \\
\alpha = \tfrac{9}{62} \quad \beta = -\tfrac{63}{62} \quad \gamma = \tfrac{17}{62} \qquad & \kappa = \tfrac{3}{10} \quad \mu = -\tfrac{3}{4} \quad \nu = \tfrac{1}{20}
\end{aligned}
$$

Derivative

Interpolation

Coefficients