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Under-resolved LES of Rayleigh-Benard convection; effects of Prandtl number anisotropy

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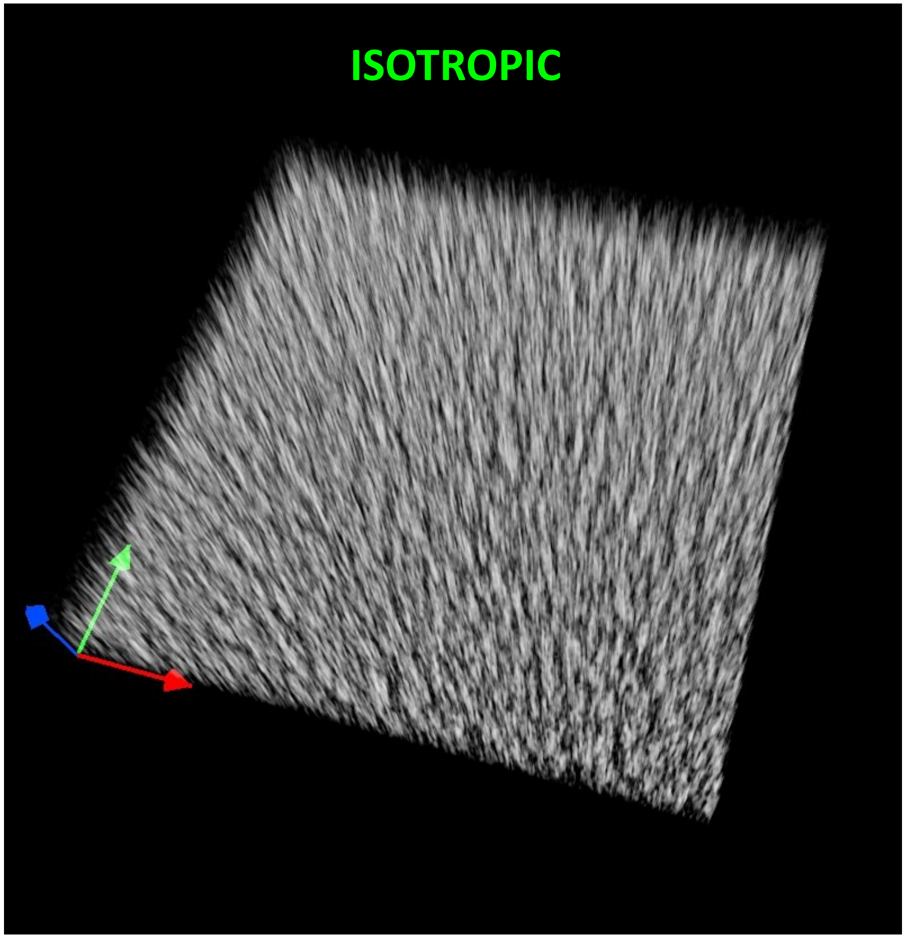
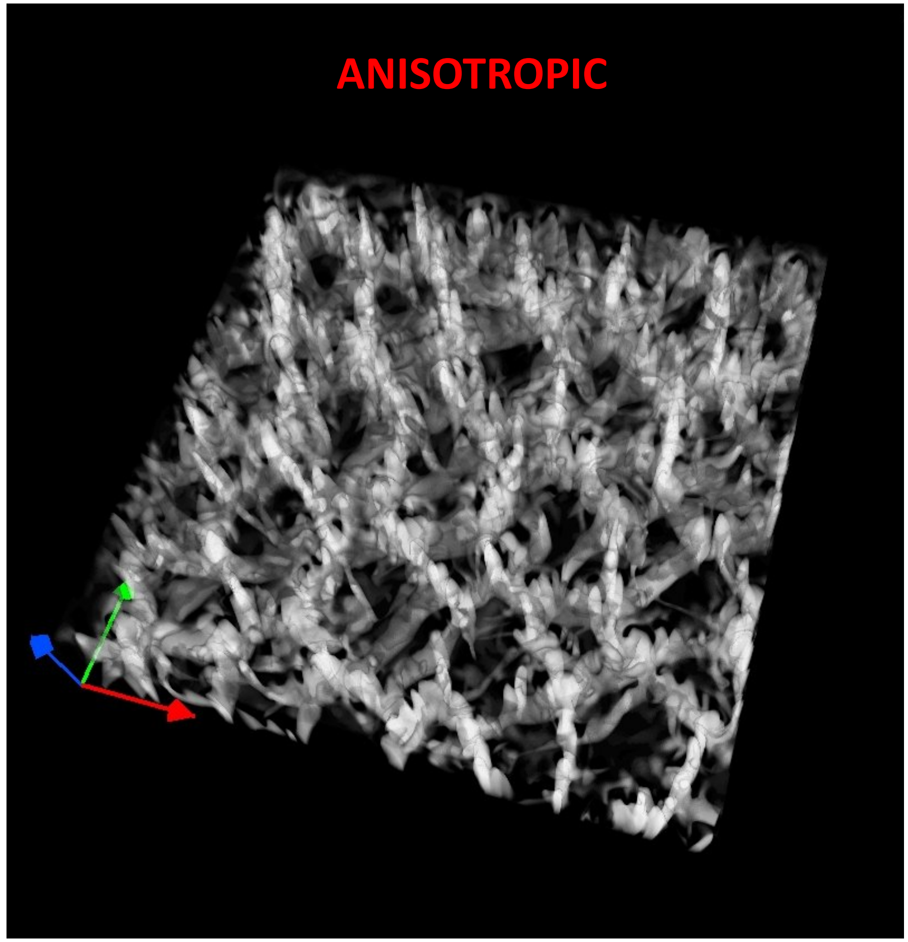
Convection over heated plane – effects of viscosity anisotropy
(separate effective viscosity attributed to the horizontal and vertical direction)

Piotrowski et al, "On numerical realizability of thermal convection", JCP, Vol. 228, 2009





$\nu_v=\kappa_v=2.5\ m^2s^{-1}$
$\nu_h=\kappa_h=2.5\ m^2s^{-1}$
$r_\nu=r_\kappa=1$

No mean wind, heatflux 20 Kms^{-1},
dx=dy=500 m, dz=50 m,
128x128x181 gridpoints

$\nu_v=\kappa_v=2.5\ m^2s^{-1}$
$\nu_h=\kappa_h=70\ m^2s^{-1}$
$r_\nu=r_\kappa=3.6e-2$

Rayleigh number in underresolved simulations

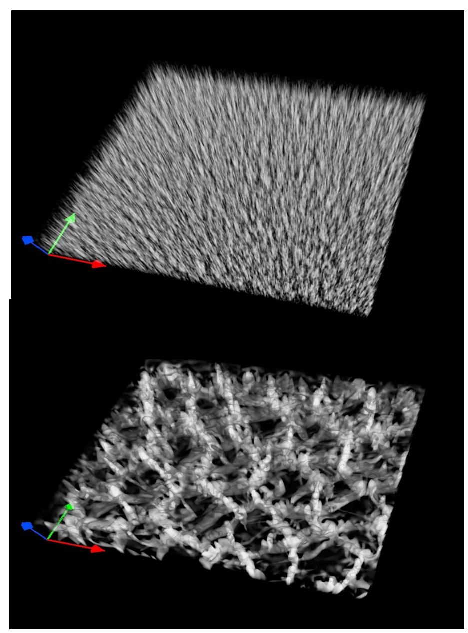
$$Ra = \frac{g\Delta\overline{\theta}h^3}{\overline{\theta}\nu\nu_\theta}$$

g – gravity acceleration
h – fluid layer thickness
ν – effective viscosity
ν_θ – effective diffusivity (=κ)
Δθ /θ – pot. temperature, relative change over h

Ra measures relative magnitude of buoyancy and viscous forces

rigid/stress-free
lower/upper

Ra_c=1100.657



>> critical

≈ critical

Linear theory extension – admitting full set of effective stress tensor

(separate effective viscosity attributed to horizontal and vertical direction AND each momentum equation)

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\phi + g\alpha\theta\nabla z + \boldsymbol{\Delta}\circ\mathbf{u} \ ,$$

$$\frac{\partial \theta}{\partial t} = \beta w + \kappa_h \partial_h^2\theta + \kappa_v \partial_z^2\theta \ ,$$

$$\nabla\cdot\mathbf{u} = 0 \ ,$$

Prandtl number anisotropy – e.g. disparate approximations to momentum equations (full set of stress tensor entries)

$$\boldsymbol{\Delta} := (\widehat{\nu}_h\partial^2 + \triangle_0,\ \widehat{\nu}_h\partial^2 + \triangle_0,\ \widehat{\nu}_v\partial^2 + \triangle_0)$$

$$\triangle_0 := \nu_h\partial_h^2 + \nu_v\partial_z^2 \ , \quad \partial_h^2 := \partial_x^2 + \partial_y^2 \ ,$$

Anisotropic viscosity (coefficients at diagonal entries of stress tensor)

Applying operator of rotation to momentum equations:

$$\frac{\partial}{\partial t}(\nabla \times \mathbf{u}) = g\alpha\nabla \times \theta\nabla z + \boxed{\triangle_0 \nabla \times \mathbf{u}} + \boxed{\triangle \nabla \times (\widehat{\nu} \circ \mathbf{u})}$$

This term describes possible production of baroclinic vorticity

Taking rotation once again and considering the vertical component:

$$\frac{\partial}{\partial t}\partial^2 w = g\alpha\partial_h^2\theta + \boxed{\triangle_0 \partial^2 w} + \boxed{(\widehat{\nu}_v \partial_h^2 + \widehat{\nu}_h \partial_z^2)\partial^2 w}$$

Equation set for vertical velocity and potential temperature becomes:

$$\left(\frac{d^2}{dz^2} - k^2\right)\left((\widehat{\nu}_h + \nu_v)\frac{d^2}{dz^2} - (\widehat{\nu}_v + \nu_h)\,k^2 - p\right)\hat{w} = g\alpha k^2\hat{\theta}$$

$$\left(\kappa_v \frac{d^2}{dz^2} - \kappa_h k^2 - p\right)\hat{\theta} = -\beta\hat{w}\,.$$

Assuming solution in Fourier modes:

$$w = \hat{w}(z)\exp[i(k_x x + k_y y) + pt] \ ,$$
$$\theta = \hat{\theta}(z)\exp[i(k_x x + k_y y) + pt] \ ; \quad k^2 := k_x^2 + k_y^2 \ , \quad i := \sqrt{-1}$$

$$\left(\frac{d^2}{dz^2} - k^2\right)\left((\widehat{\nu}_h + \nu_v)\frac{d^2}{dz^2} - (\widehat{\nu}_v + \nu_h)\,k^2 - p\right)\hat{w} = g\alpha k^2 \hat{\theta}$$

$$\left(\kappa_v \frac{d^2}{dz^2} - \kappa_h k^2 - p\right)\hat{\theta} = -\beta\hat{w} \, .$$

Note that number of parameters is now effectively reduced.

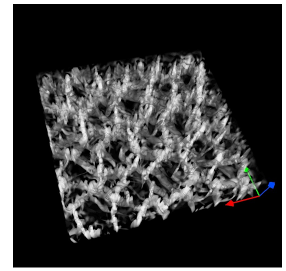
$$\left(\frac{d^2}{dz^2} - k^2\right)\left(\nu_{veff}\frac{d^2}{dz^2} - \nu_{heff}k^2 - p\right)\left(\kappa_v \frac{d^2}{dz^2} - \kappa_h k^2 - p\right)\hat{w}$$
$$= -g\alpha k^2 \beta \hat{w} \, .$$

Possible stress tensor realizations - two simple examples

1. Prandtl number isotropy - anisotropic filtering of model equations

$$\nu_v = \kappa_v = x \ m^2 s^{-1}$$
$$\nu_h = \kappa_h >> \nu_v = \kappa_v$$

$$\hat{\nu}_v = \hat{\nu}_h = 0$$



2. Prandtl number anisotropy –anisotropic filtering of either momentum equations or temperature equation

$$\kappa_v = x \ m^2 s^{-1}$$
$$\kappa_h >> \kappa_v$$
$$\nu_v = \nu_h = x \ m^2 s^{-1}$$

$$\hat{\nu}_v = \hat{\nu}_h = 0$$

… or in terms of extended linear theory

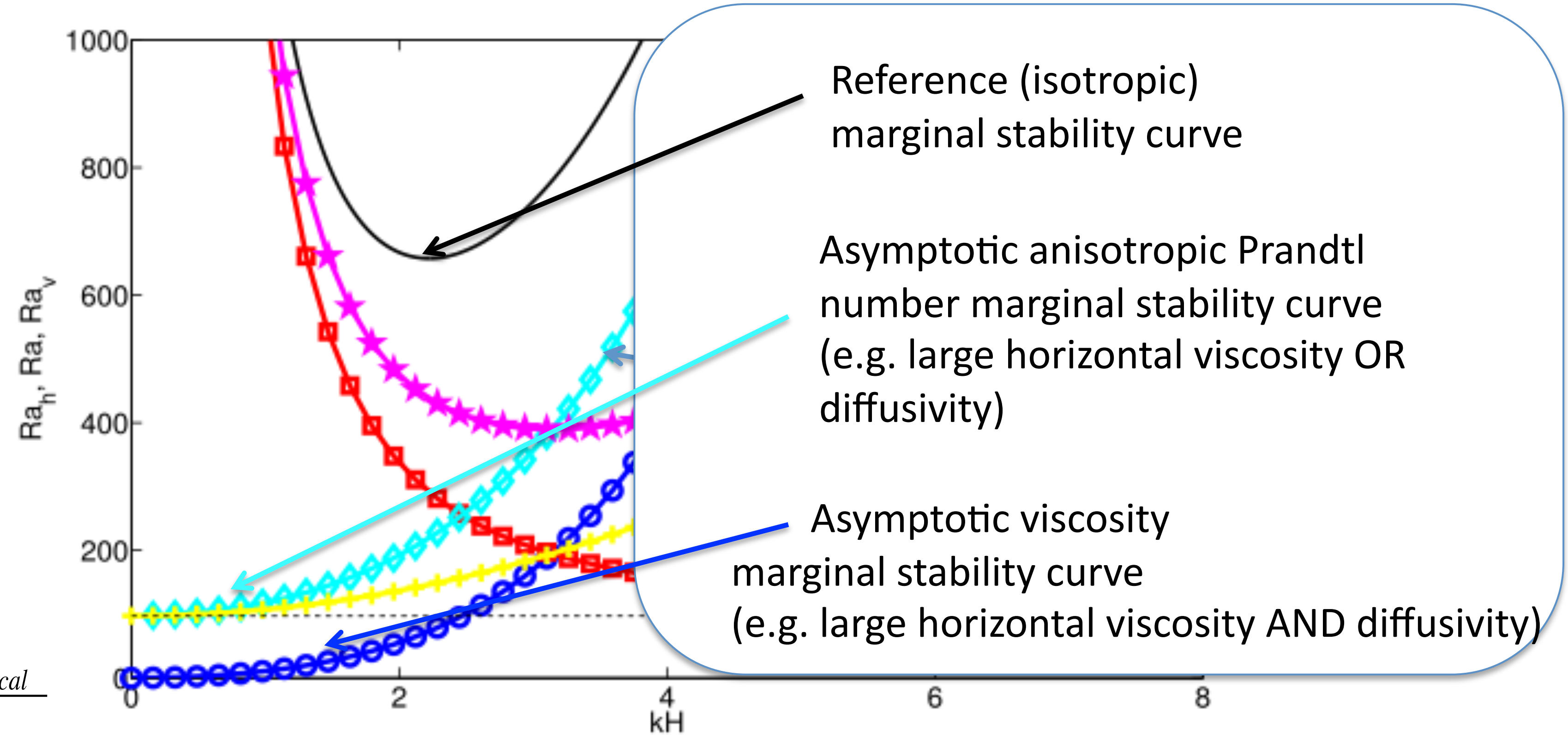
$$\kappa_v = x \ m^2 s^{-1}$$
$$\kappa_h >> \kappa_v$$
$$\nu_v = \nu_h = 0$$

$$\hat{\nu}_v = \hat{\nu}_h = x \ m^2 s^{-1}$$

Numerical substantiation to follow ….

Example 1. refers to the "blue circle" asymptote

Example 2. refers to the "cyan diamond" asymptote

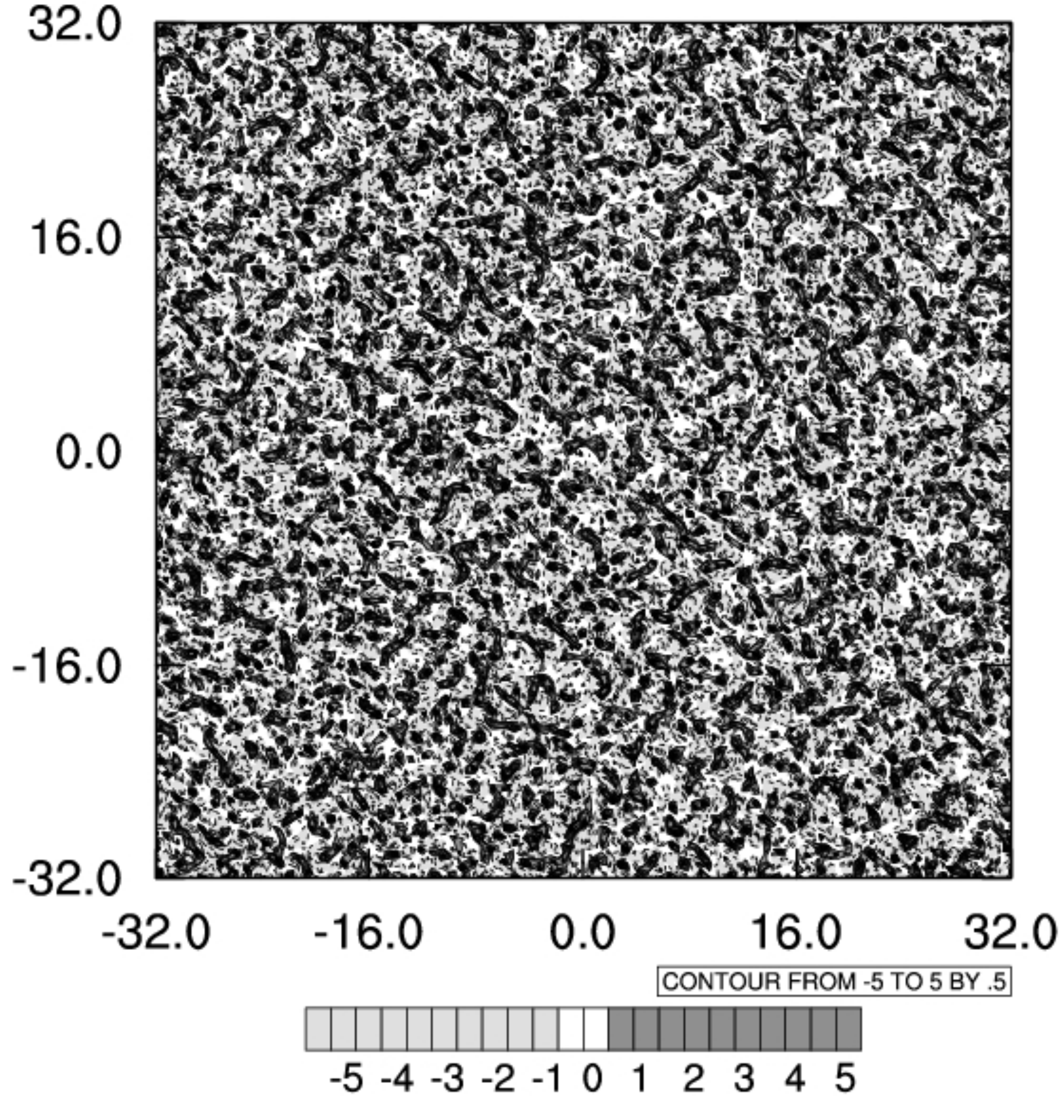


$$r_v = \frac{v_{vertical}}{v_{horizontal}}$$

Fig. 1. Asymptotic marginal stability relations for viscosities $\nu_h = \nu_v$ and thermal diffusivities $\kappa_h = \kappa_v$ (solid), viscosity anisotropy ratios $r_{\nu,\kappa} = 0$ (blue circles), $r_{\nu,\kappa} = \infty$ (red squares), $r_\nu = 0, r_\kappa = 1$ (cyan diamonds), $r_\nu = \infty, r_\kappa = 1$ (magenta stars), $r_\nu = \infty, r_\kappa = 0$ (yellow plus sign); here h and v denote respective values in the horizontal and the vertical. Corresponding Rayleigh numbers Ra are shown as functions of the non-dimensional horizontal wave number kH. For each curve the stability region lies beneath. The square, diamond and plus-sign asymptotics tend to the π^4 limit.

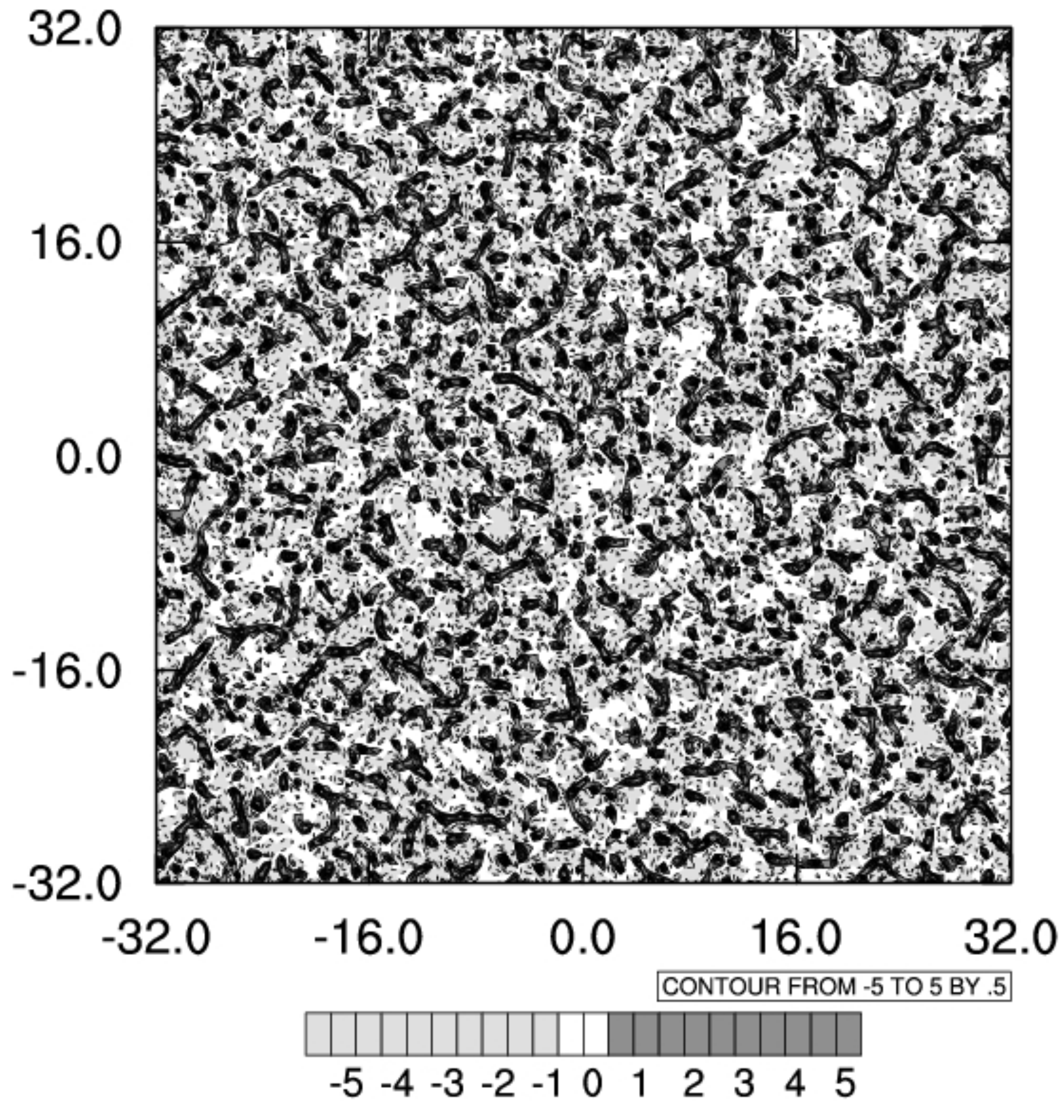
Convection over heated plane, heatflux 20 Kms^{-1}, dx=dy=125 m, dz = 50 m, 512x512x180 gridpoints

Reference Implicit LES solution after 4h of simulated time at 1/3 of the boundary layer depth.



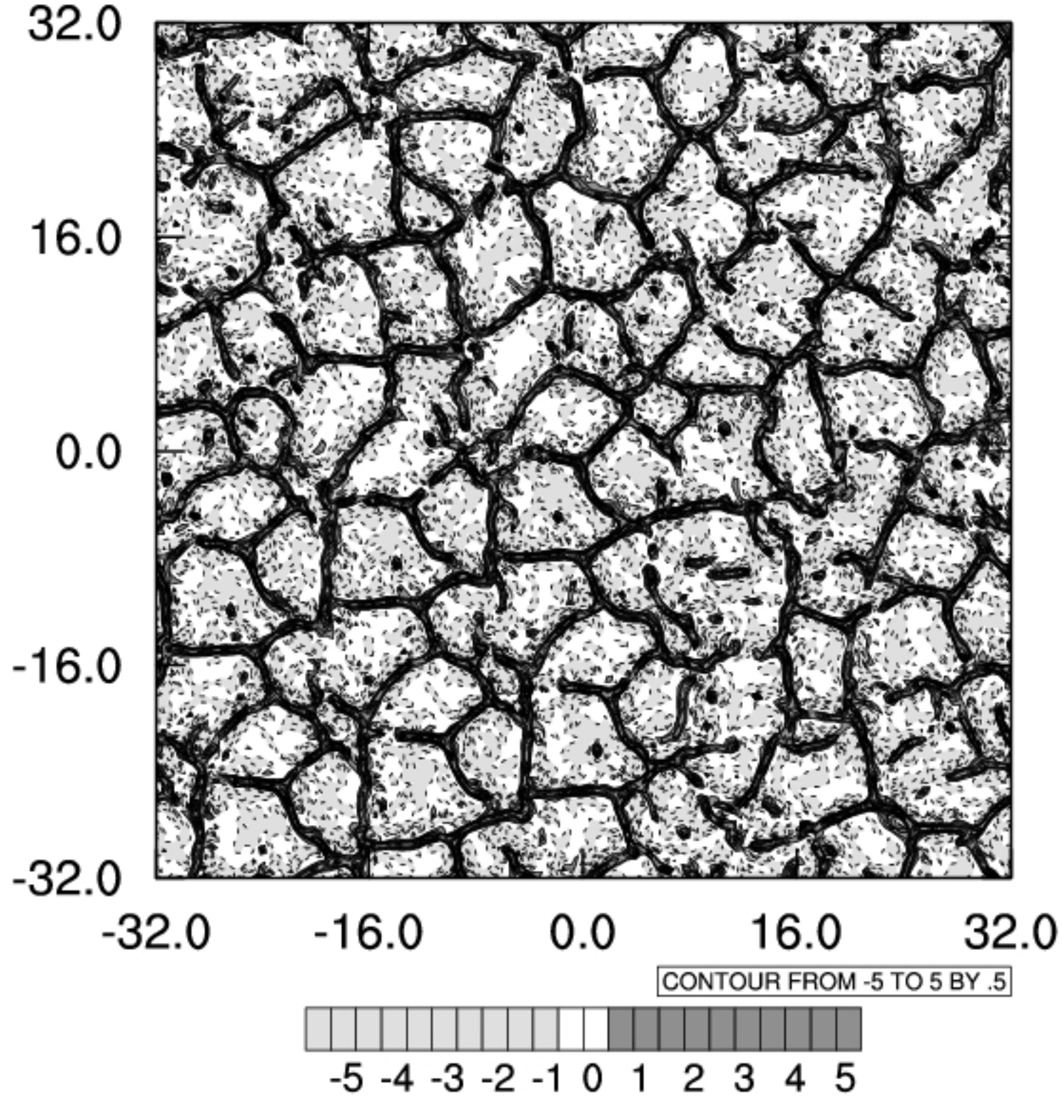
Convection over heated plane, heatflux 20 Kms^{-1}, dx=dy=125 m, dz = 50 m, 512x512x180 gridpoints

Illustration to example 1: anisotropic viscosity $r_\nu = r_\kappa = 8e{-2}$.



Convection over heated plane, heatflux 20 Kms^{-1}, dx=dy=125 m, dz = 50 m, 512x512x180 gridpoints

Illustration to example 2: Prandtl number anisotropy $Pr_v : Pr_h = 1 : 6e\text{-}3$. The same Rayleigh number as in example 1.



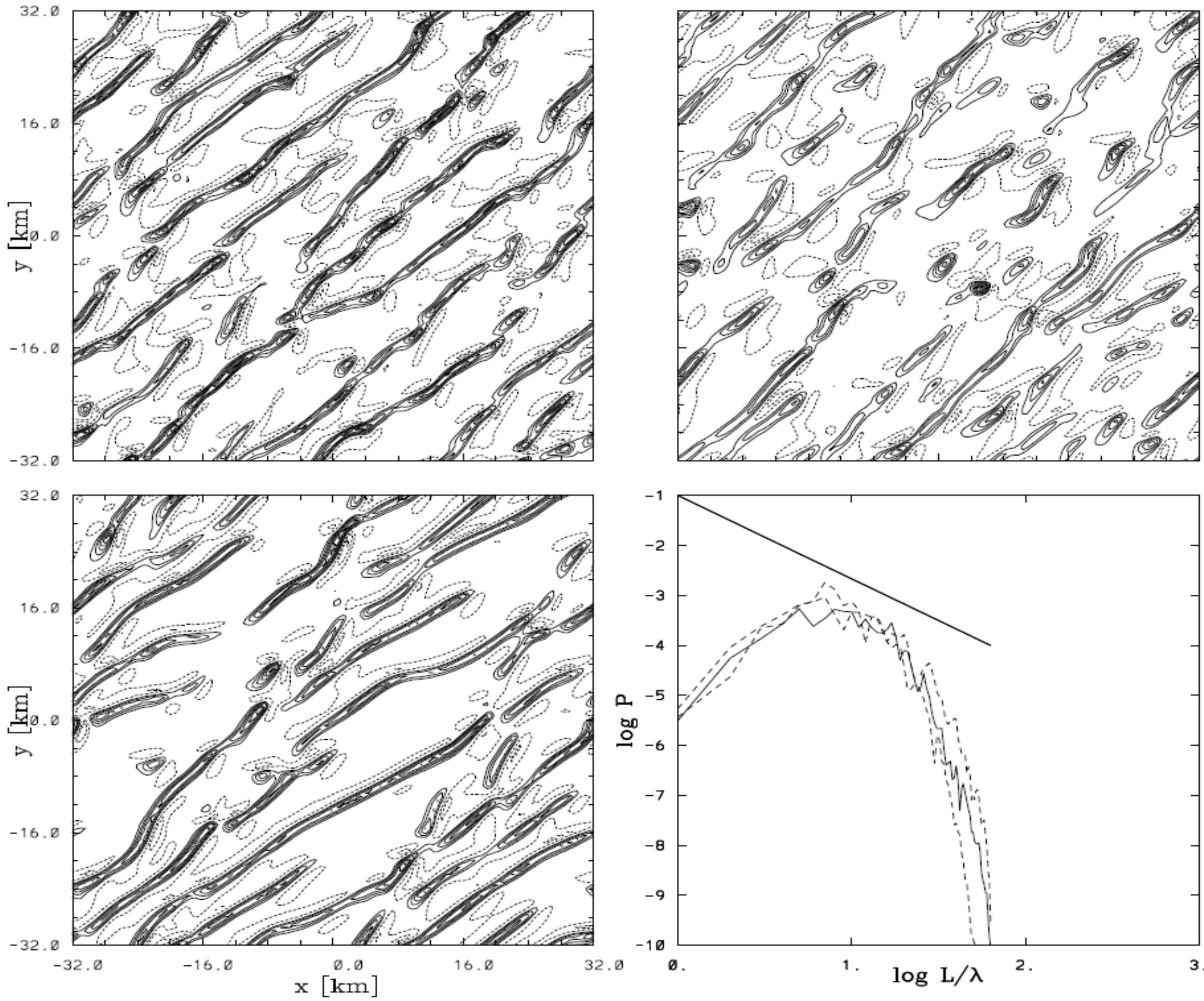
Conclusions

- Anisotropic viscosity and Prandtl number anisotropy can modify marginal stability and mode growth rates of realized R-B convection.
- Prandtl number anisotropy effects may alter convective picture at much higher Rayleigh number than anisotropic viscosity effects alone, due to a modification of the mode stability range and growth rate change.
- There may be an option to use the derived linear theory for tune a numerical model for specific task.

Convective picture – reference ILES simulations, but with different filters (anisotropic viscosity)

Composite MPDATA: 1st order UPWIND every 4th dt

1-2-1 filter



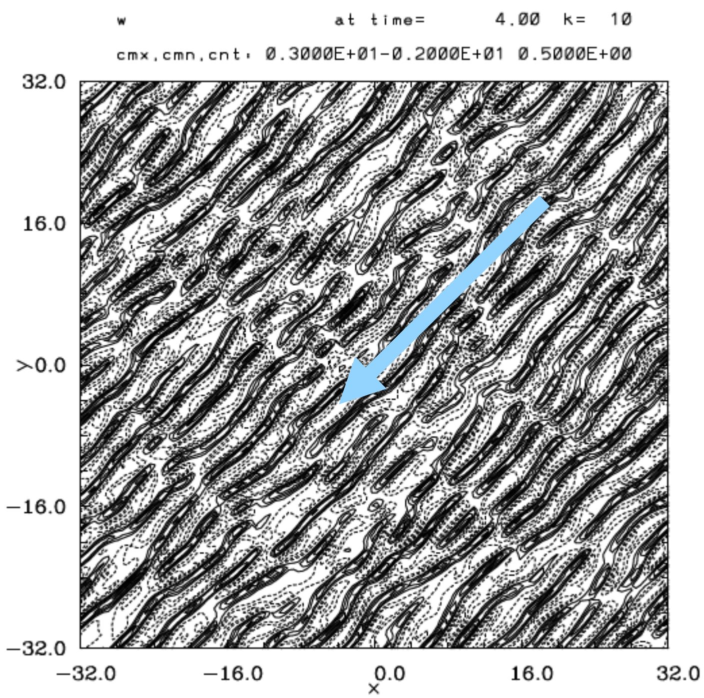
Explicit anisotropic viscosity

Diagonal 2D Spectra

Different filtering gives similar results

Example of numerical substantiation

Series of LES and ILES using the EULAG model



dz=50 m

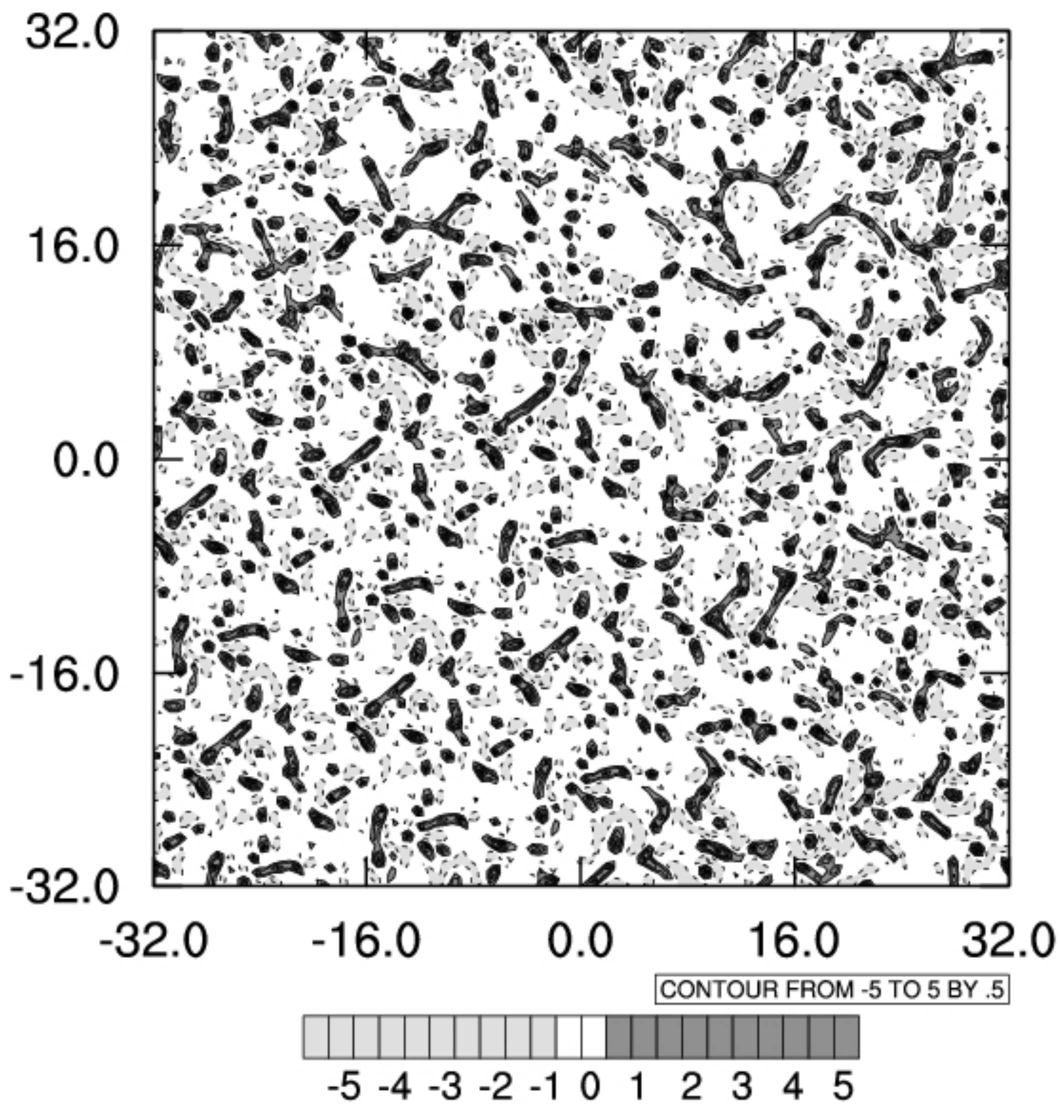
V = [-10,-10] m/s

dx=dy=500 m

Heat flux
hfx≈200 W/m^2

Flat lower boundary, doubly periodic horizontal domain, Boussinesq option

Reference setup alludes to contemporary, mesoscale cloud-resolving NWP


32.0
16.0
0.0
-16.0
-32.0
-32.0
-16.0
0.0
16.0
32.0
CONTOUR FROM -5 TO 5 BY .5
-5 -4 -3 -2 -1 0 1 2 3 4 5