Wave Dispersion Properties of Mixed Finite Elements

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Overview

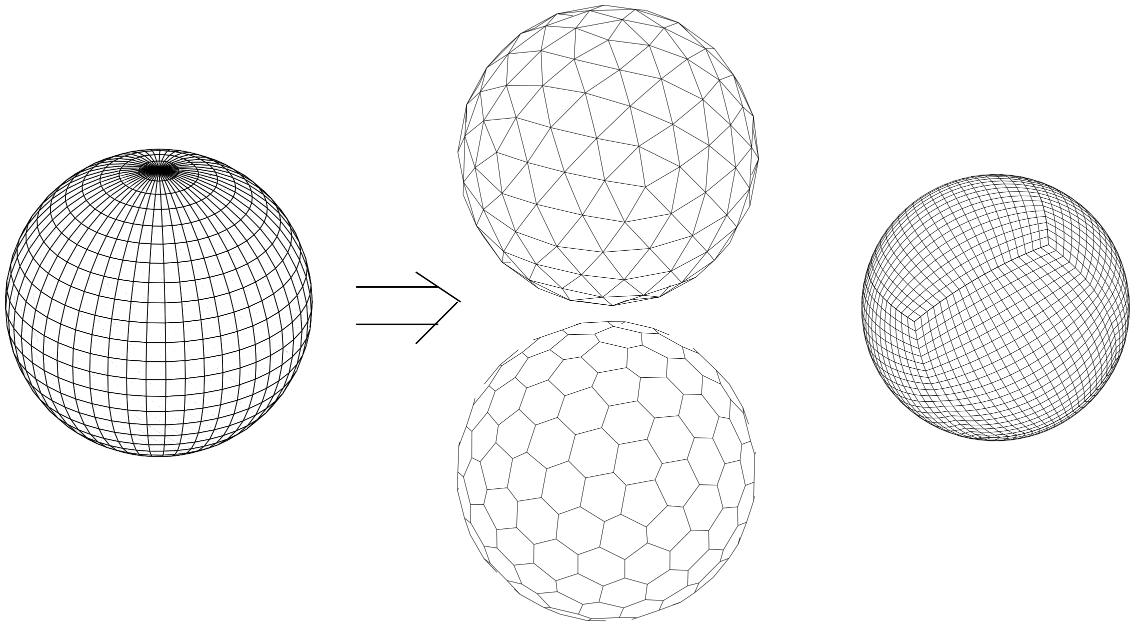
- Gung-Ho dynamical core project
- Desirable properties of a dynamical core
- Mixed finite elements
- Dispersion properties + mass lumping
- Simulations
- Conclusions + Future Work

Gung-Ho

- Globally Uniform Next Generation-Highly Optimised
- 5 year joint UKMO, NERC (UK universities) and STFC research project for next generation dynamical core
- A unified approach for high-resolution NWP (O (10) km global, O (100) m locally) to low-resolution (around 300 km) climate applications.
- Scalable code out to O(10^5 - 10^6) processors.
- Inherent conservation of mass.
- At least comparable accuracy to contemporary dynamical core in same clock time.
- A regional modelling capability, either or both of variable grid, or limited-area driven by boundary conditions.

Horizontal Grid

- Met Office UM uses a lat-lon grid
    - Significant resolution clustering near polar singularity:
    - $\Delta\lambda = 25\text{km} \rightarrow \Delta\lambda^{\text{pole}} = 75\text{m},$
    - $\Delta\lambda = 10\text{km} \rightarrow \Delta\lambda^{\text{pole}} = 12\text{m}.$



Discretisation properties

Staniforth and Thuburn [QJ **138** (2012)] list a number of essential and desirable properties for an atmospheric model

- Conservation of mass
- Accurate representation of balance and adjustment
- Absence of or well controlled computational modes
- Geopotential and pressure gradients are not a source of unphysical vorticity
- Pressure terms are energy conserving
- Coriolis terms are energy conserving
- No spurious propagation of Rossby modes
- Conservation of axial angular momentum
- Accuracy approaching second order
- Minimal grid imprinting

Mixed Finite Elements

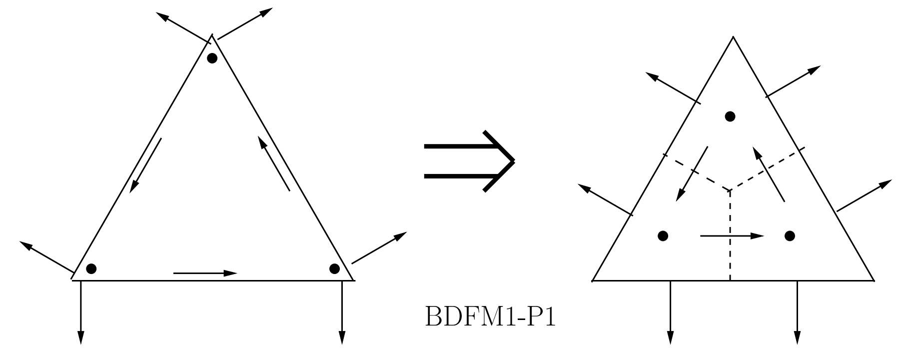
Cotter and Shipton [JCP **232** (2012)] propose some families of mixed finite element discretisations that retain a number of these properties

- energy conservation
- mass conservation
- absence of spurious pressure modes
- steady geostrophic modes on an f plane
- flexibility to adjust the balance of velocity degrees of freedom to pressure ones
- Use different function spaces to obtain C-grid like staggering

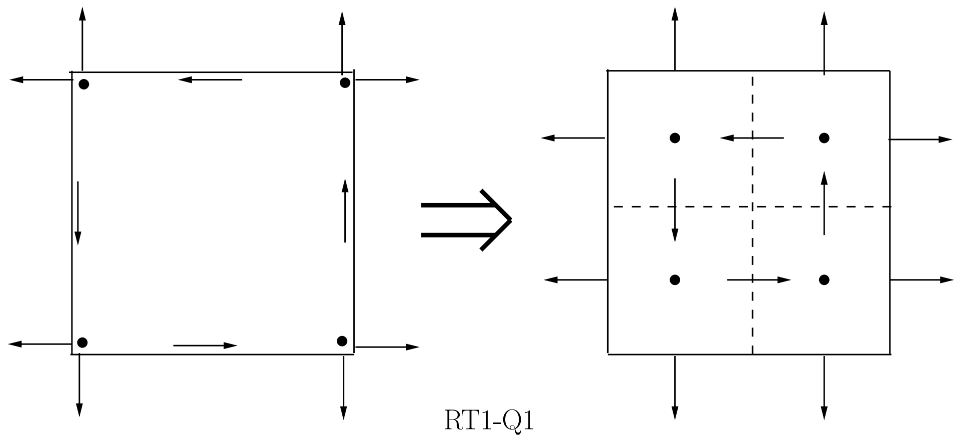
$$\begin{tikzcd} \mathbb{V}_0 \arrow[r, "\nabla^{\perp}"] & \mathbb{V}_1 \arrow[r, "\nabla."] \arrow[l, dashed, bend left, "\mathbf{k}.\tilde{\nabla}\times"] & \mathbb{V}_2 \arrow[l, dashed, bend left, "\tilde{\nabla}"] \end{tikzcd}$$

Mixed Finite Elements

- CS12 propose RTk-Qk family of elements on quads or BDFM1-P1 on triangular elements.
- RT0-Q0 is finite element analogue of C-grid finite difference.



BDFM1-P1



RT1-Q1

Discretised Equations

- Linear SWE:

$$
\boxed{
\begin{aligned}
\frac{\partial \Phi}{\partial t} + \Phi_0 \nabla . \mathbf{u} = 0 \\
\frac{\partial \mathbf{u}}{\partial t} + \nabla \Phi + f \mathbf{u}^{\perp} = 0
\end{aligned}
}
$$

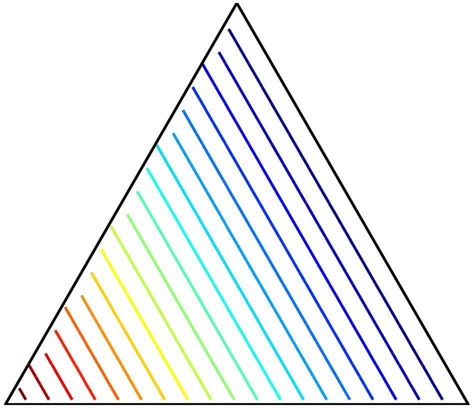
Discretised Equations

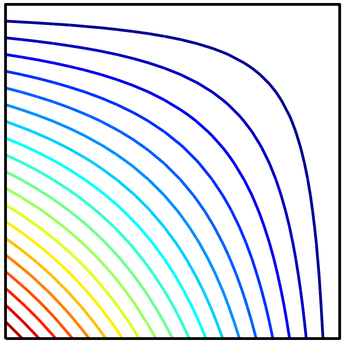
- Weak form:

$$\int_\Omega \rho \frac{\partial \Phi}{\partial t} + \Phi_0 \rho \nabla . \mathbf{u} dA = 0$$

$$\int_\Omega \mathbf{w} . \frac{\partial \mathbf{u}}{\partial t} - (\nabla . \mathbf{w})\, \Phi + f \mathbf{w} . \mathbf{u}^\perp dA = 0$$

- Pressure basis (ρ): Linear, discontinuous



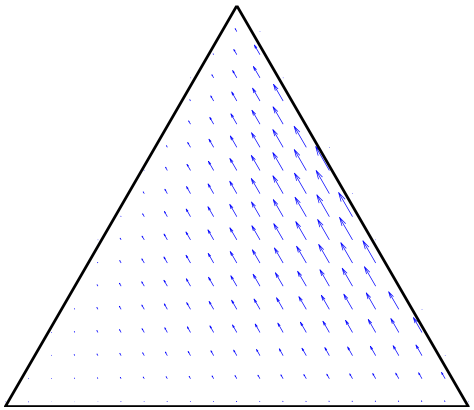


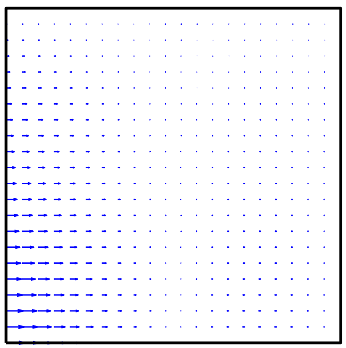
Discretised Equations

- Discretised as:

$$M_\Phi \frac{\partial \Phi}{\partial t} + \Phi_0 D_2 \mathbf{u} = 0$$
$$M_u \frac{\partial \mathbf{u}}{\partial t} + D_1 \Phi + F\mathbf{u} = 0$$

- Velocity basis $(\mathbf{w})$: Quadratic, continuous normal components





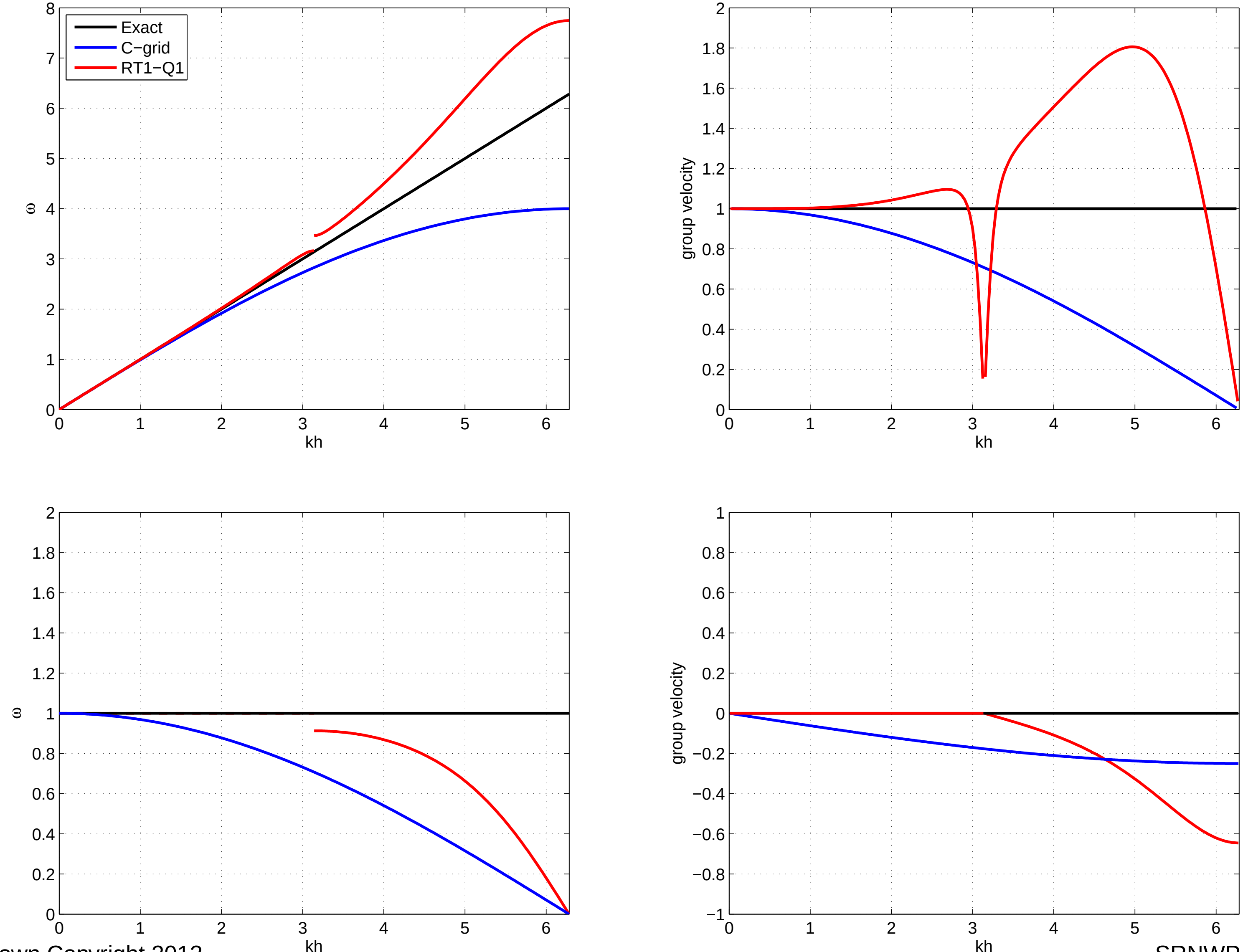
Dispersion Analysis

- Seek solutions of the form:

$$u(\mathbf{x},t) = U_{\mathbf{k}} \exp\left(i\left[\mathbf{k}.\mathbf{x} - \omega t\right]\right),$$
$$\Phi(\mathbf{x},t) = P_{\mathbf{k}} \exp\left(i\left[\mathbf{k}.\mathbf{x} - \omega t\right]\right)$$

- Solve resultant dispersion problem $\det\left[\mathrm{A}\left(\mathbf{k},\omega\right)\right] = \mathbf{0}$,

1D Dispersion Properties



Mass lumping

- At problematic scale ($kh = \pi$) there are two solutions

$$\omega^2_{kh=\pi} = \frac{5}{2}\left[\frac{\Phi_0}{(h/2)^2} + \frac{f^2}{3}\right],$$
$$\omega^2_{kh=\pi} = 3\left[\frac{\Phi_0}{(h/2)^2} + \frac{f^2}{3}\right].$$

- Solution: introduce partial mass-lumping into M_u

$$M_u = \frac{1}{3}\begin{bmatrix} \frac{4}{10} & \frac{1}{5} & -\frac{1}{10} \\ \frac{1}{5} & \frac{4}{5} & \frac{1}{5} \\ -\frac{1}{10} & \frac{1}{5} & \frac{4}{10} \end{bmatrix} \Longrightarrow \frac{1}{3}\begin{bmatrix} \frac{3}{10} - \alpha & \frac{1}{5} & \alpha \\ \frac{1}{5} & \frac{4}{5} & \frac{1}{5} \\ \alpha & \frac{1}{5} & \frac{3}{10} - \alpha \end{bmatrix}$$

Mass lumping

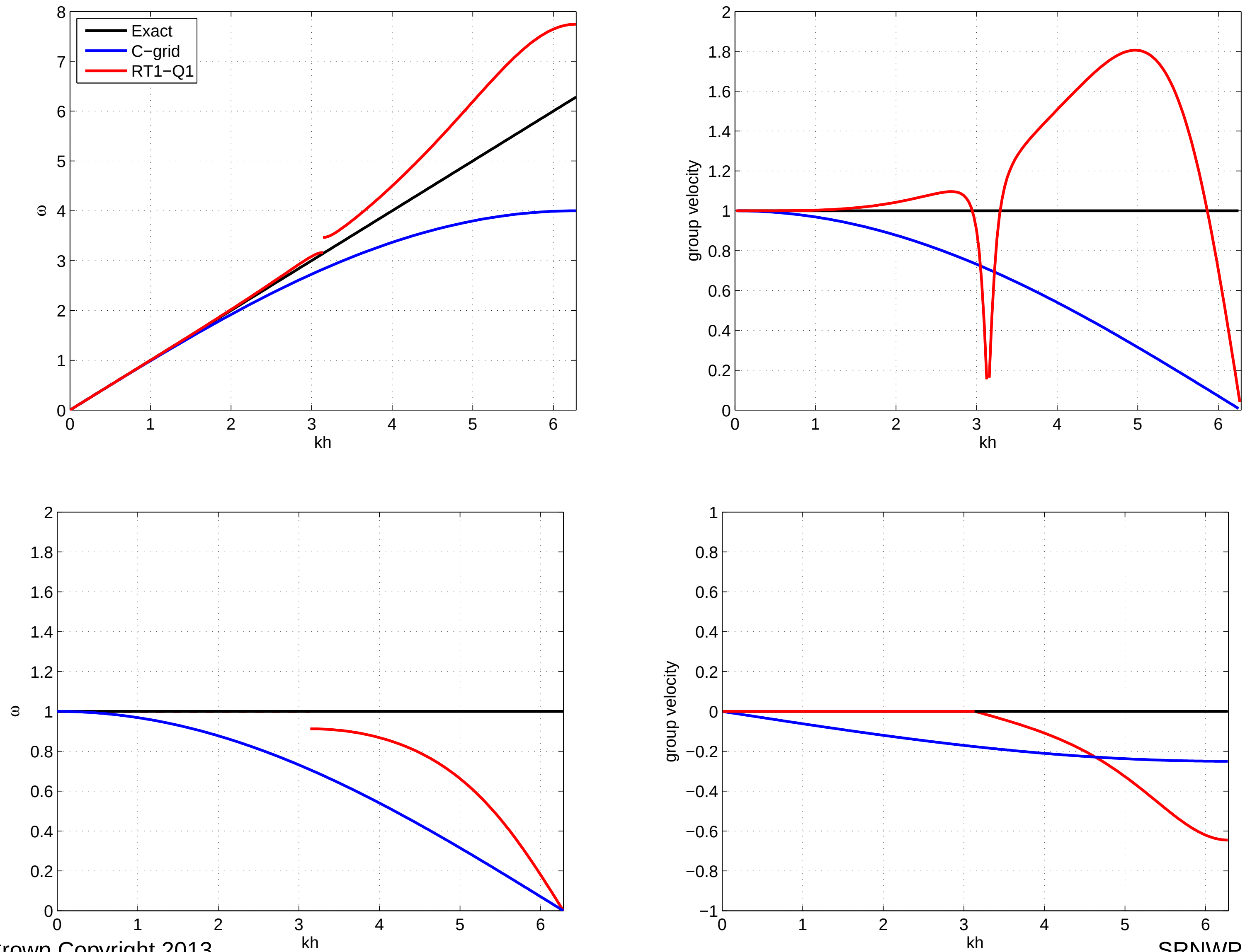
- Setting $\alpha = -3/20$ both solutions coincide

$$\omega^2_{kh=\pi} = \frac{5}{2}\left[\frac{\Phi_0}{(h/2)^2} + \frac{f^2}{3}\right],$$
$$\omega^2_{kh=\pi} = \frac{3}{3/5 - 4\alpha}\left[\frac{\Phi_0}{(h/2)^2} + \frac{f^2}{3}\right].$$

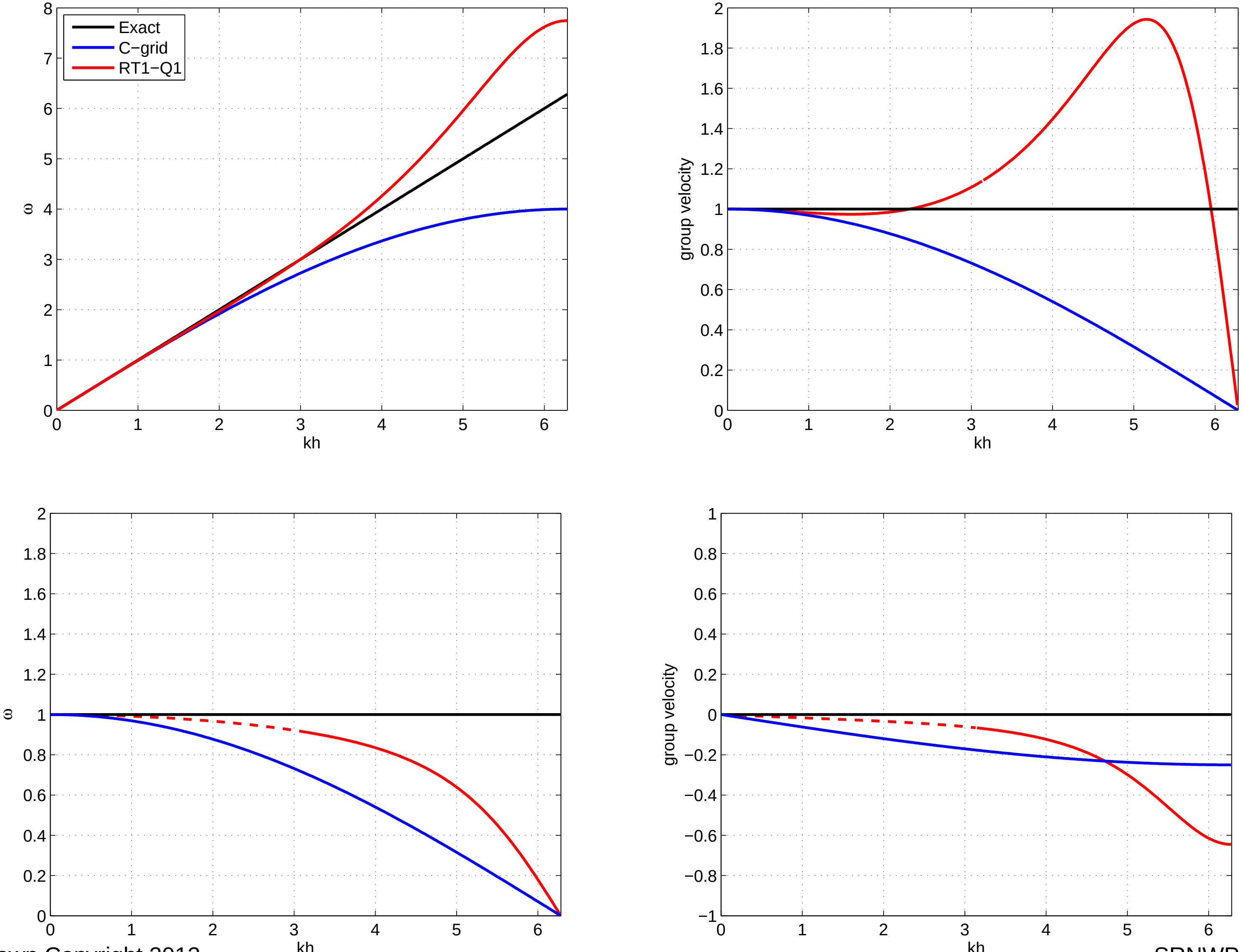
- Solution: introduce partial mass-lumping into M_u

$$M_u = \frac{1}{3}\begin{bmatrix} \frac{4}{10} & \frac{1}{5} & -\frac{1}{10} \\ \frac{1}{5} & \frac{4}{5} & \frac{1}{5} \\ -\frac{1}{10} & \frac{1}{5} & \frac{4}{10} \end{bmatrix} \Longrightarrow \frac{1}{3}\begin{bmatrix} \frac{3}{10} - \alpha & \frac{1}{5} & \alpha \\ \frac{1}{5} & \frac{4}{5} & \frac{1}{5} \\ \alpha & \frac{1}{5} & \frac{3}{10} - \alpha \end{bmatrix}$$

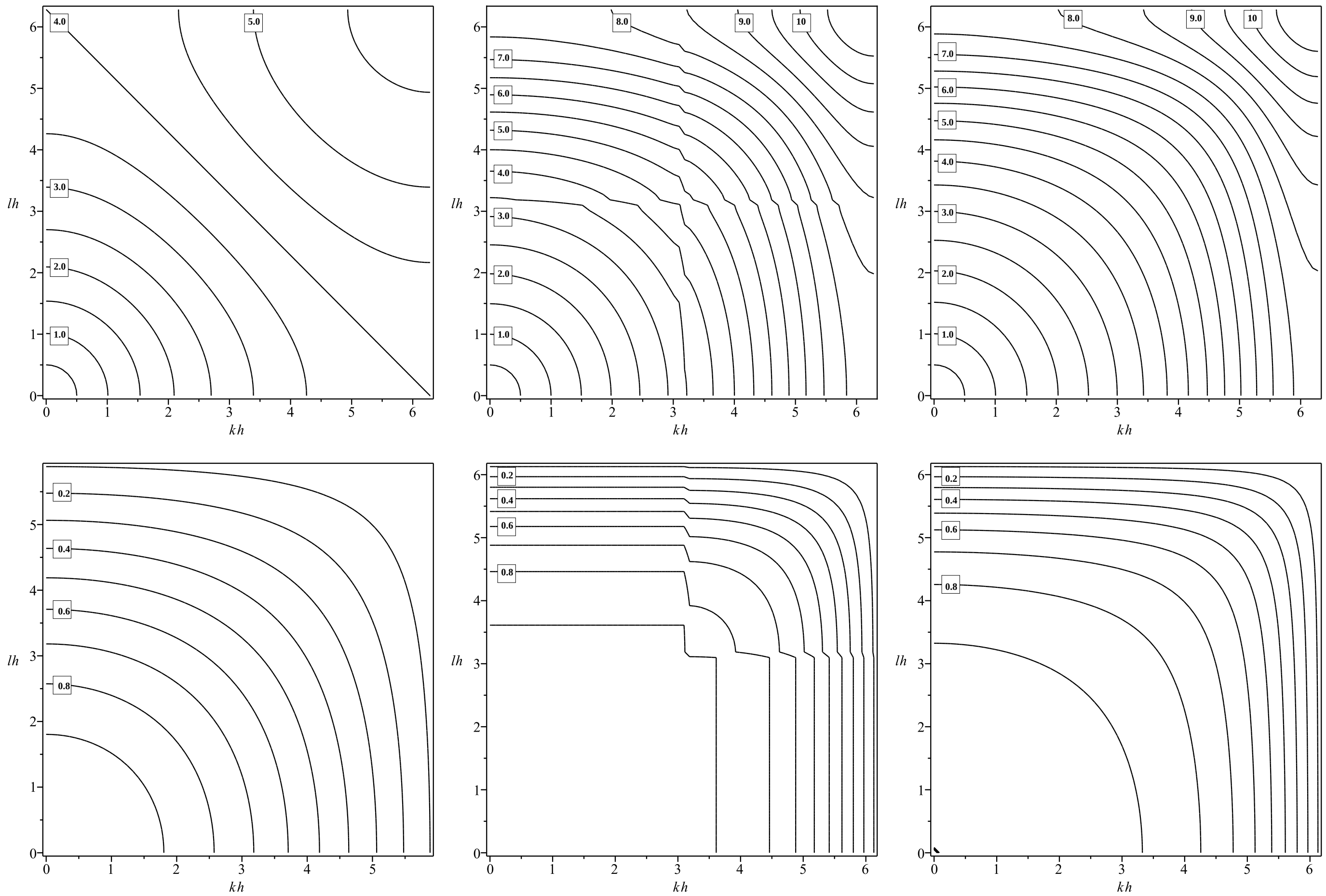
1D Dispersion Properties - (Unlumped)



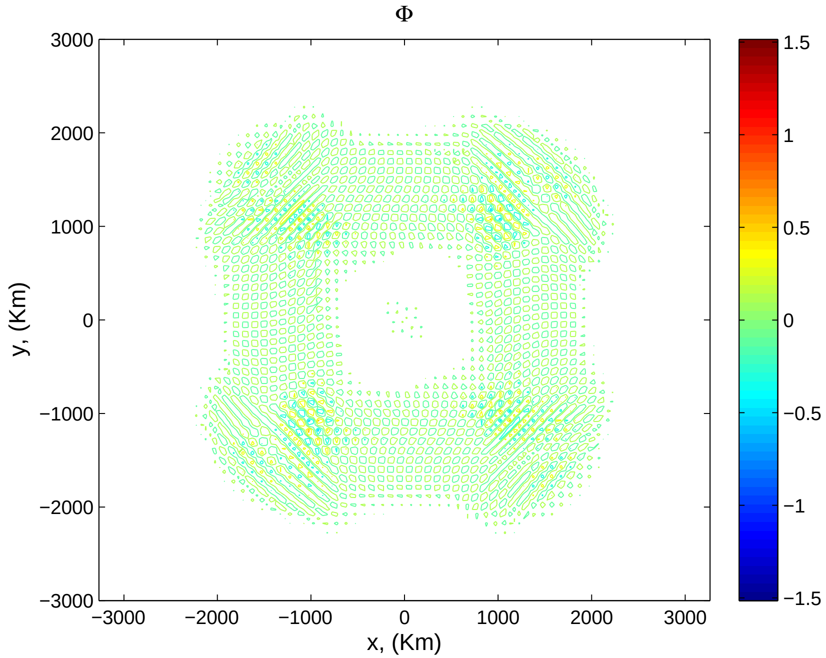
1D Dispersion Properties - (Partial-lumping)

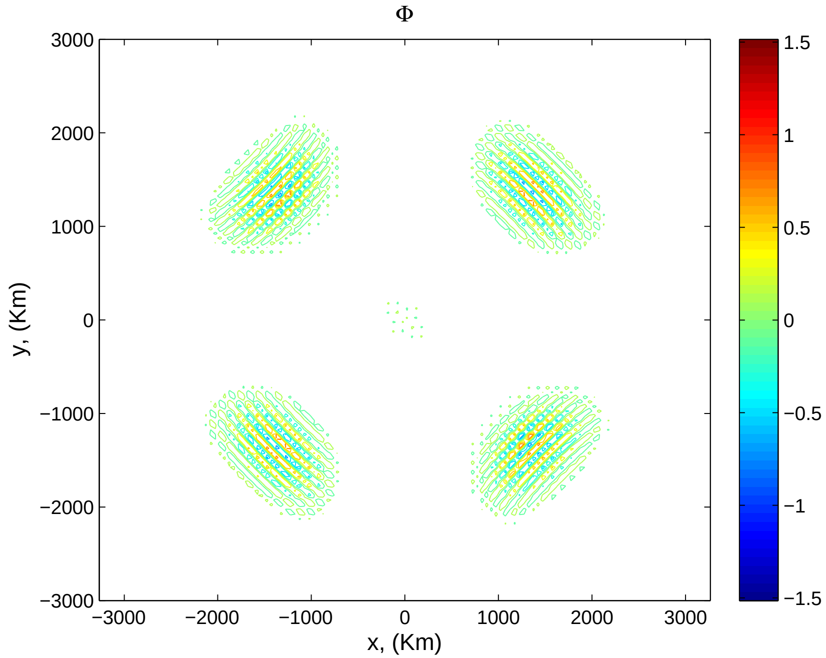


Dispersion Properties (Quads)

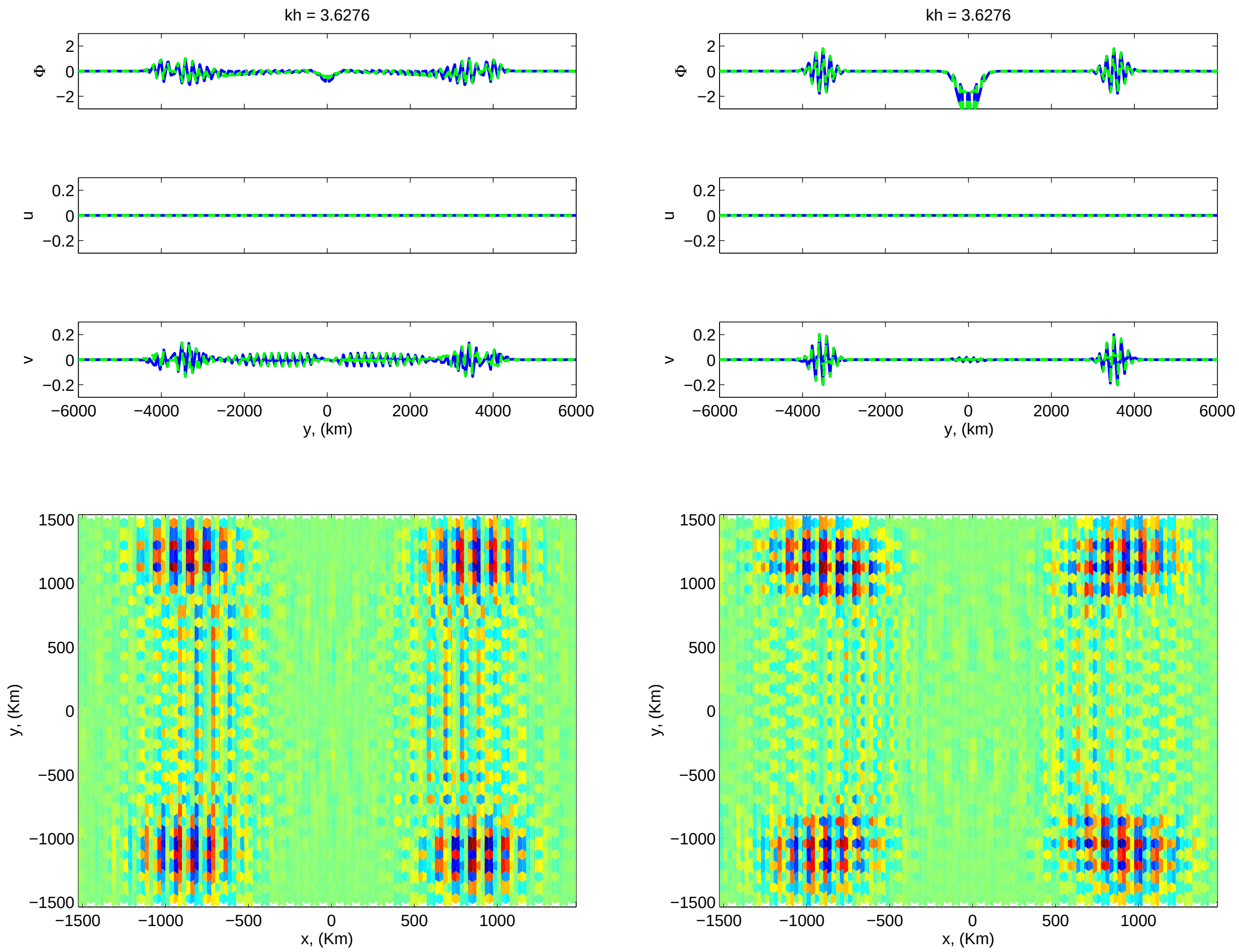


Simulations (Quadrilaterals)





Simulations (Triangles)



Conclusions

- Mixed finite elements provide a useful methodology to obtain many desired properties
- Both RT1 and BDFM1 elements have dispersion problems for element scale waves
  - This can be fixed for RT1 through partial mass-lumping
  - Some evidence this also possible for BDFM1
- Non-diagonal mass-matrix $\Longrightarrow$ global matrix inversion (even for explicit time schemes)
- Low order RT0 elements also possible for Hex, tri and quad grids
  - No gap in dispersion relation
  - Formally less accurate on non-uniform girds
  - Hexagonal + triangular grids suffer same compuational mode as C-grid equivalent

Dispersion Properties (Triangles)