An exact analytical solution for linear gravity and sound wave expansion of the compressible, non-hydrostatic Euler equations

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**Michael Baldauf, Daniel Reinert, Günther Zängl,
Slavko Brdar (DWD, Germany)**

Motivation

For the development of dynamical cores (or numerical methods in general) idealized test cases are an important evaluation tool.

- Idealized standard test cases with (at least approximated) analytic solutions:
    - stationary flow over mountains
      linear: *Queney (1947, ...), Smith (1979, ...) Adv Geophys, Baldauf (2008) COSMO-Newsl.*
      non-linear: *Long (1955) Tellus* for Boussinesq-approx. Atmosphere
    - Balanced solutions on the sphere: *Staniforth, White (2011) ASL*
    - non-stationary, linear expansion of gravity waves in a channel
      Skamarock, Klemp (1994) MWR for Boussinesq-approx. atmosphere
- most of the other idealized tests only possess 'known solutions' gained from other numerical models.

There exist even fewer analytic solutions which use <u>exactly the same equations</u> as the numerical model used, i.e. in the sense that the numerical model <u>converges to this solution</u>. One exception is given here:
linear expansion of gravity/sound waves on the sphere

Non-hydrostatic, compressible, shallow atmosphere, adiabatic, 3D Euler equations on a sphere with a rigid lid

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\frac{1}{\rho} \nabla p - g \mathbf{e}_z - 2 \mathbf{\Omega} \times \mathbf{v} \\
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho &= -\rho \nabla \cdot \mathbf{v} \\
\frac{\partial p}{\partial t} + \mathbf{v} \cdot \nabla p &= c_s^2 \left(\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho \right) \\
c_s &= \sqrt{\frac{c_p}{c_v} \frac{p}{\rho}}
\end{aligned}
$$

Boundary conditions:
$w(r=r_s) = 0$
$w(r=r_s+H) = 0$

most global models using the compressible equations should be able to exactly use these equations in the dynamical core for testing.

For an analytic solution only one further approximation is needed:
linearisation (= *controlled* approximation) around an
isothermal, steady, hydrostatic atmosphere

Solution strategy

Isothermal background state + shallow atmosphere approx.
→ Bretherton (1966) transformation
→ all coefficients of the linearized PDE system are constant

Shallow atmosphere approximation

- replace all prefactors $1/r$ → $1/r_s$
- in the divergence operator: omit the metric correction term $\sim w/r$
- apart from that all earth curvature metric terms are included
- (optional) Coriolis force: ‚global f-plane approx. on a sphere'

$$2\boldsymbol{\Omega}(\lambda,\phi) = f \cdot \boldsymbol{e}_r(\lambda,\phi), \quad f = \text{const.} \qquad \text{(and } \boldsymbol{v}_0 = 0\text{)}$$

Spectral representation of fields:

$$\psi(\lambda,\phi,r,t) = \sum_{k_z} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{\psi}_{lm}(k_z,t) \cdot Y_{lm}(\phi,\lambda) \cdot e^{ik_z z} \qquad z = r - r_s$$

Spherical harmonics: $Y_{lm}(\phi,\lambda) := N_{lm} \cdot P_{lm}(\sin\phi) \cdot e^{im\lambda}$

Time integration by Laplace transform

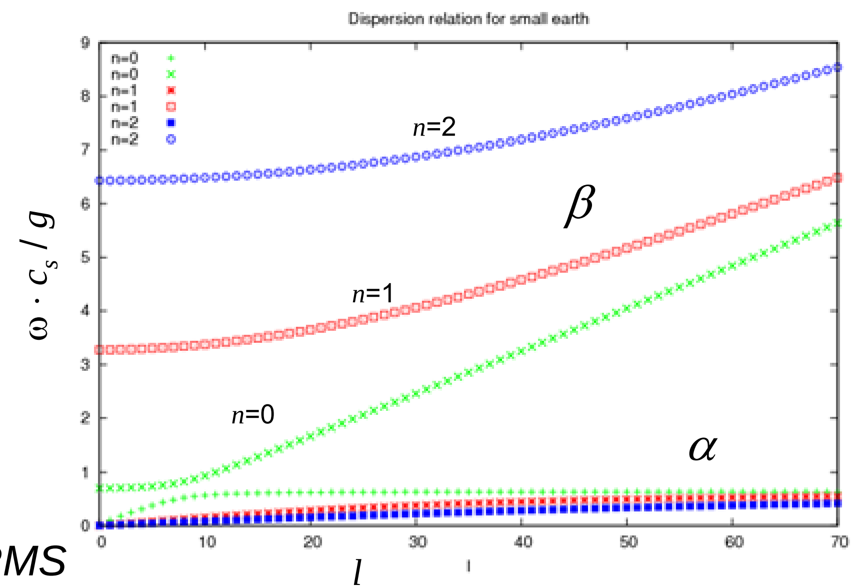
Analytic solution

for the vertical velocity w (Fourier component with k_z, spherical harmonic with l,m)

$$\hat{w}_{lm}(k_z,t) = -\frac{1}{\beta^2-\alpha^2}\left[-\alpha\sin\alpha t + \beta\sin\beta t + \left(f^2 + c_s^2\frac{l(l+1)}{r_s^2}\right)\left(\frac{1}{\alpha}\sin\alpha t - \frac{1}{\beta}\sin\beta t\right)\right] g\frac{\hat{\rho}_{lm}(k_z,t=0)}{\rho_s}$$

analogous expressions for $\hat{u}_{lm}(k_z, t)$, ...

The frequencies α, β are the gravity wave and acoustic branch, respectively, of the dispersion relation for compressible waves in a spherical channel of height H; $k_z = (\pi / H) \cdot n$



Baldauf, Reinert, Zängl (2013) subm. to QJRMS

Test scenarios

(A)Only gravity wave and sound wave expansion

(B)Additional Coriolis force (‚global f-plane approx. on a sphere‘)

$$2\boldsymbol{\Omega}(\lambda,\phi) = f \cdot \boldsymbol{e}_r(\lambda,\phi), \quad f\text{=const.} \qquad \text{(and } \boldsymbol{v}_0 = 0\text{)}$$

→ test proper discretization of inertia-gravity modes, e.g. in a C-grid discretization.

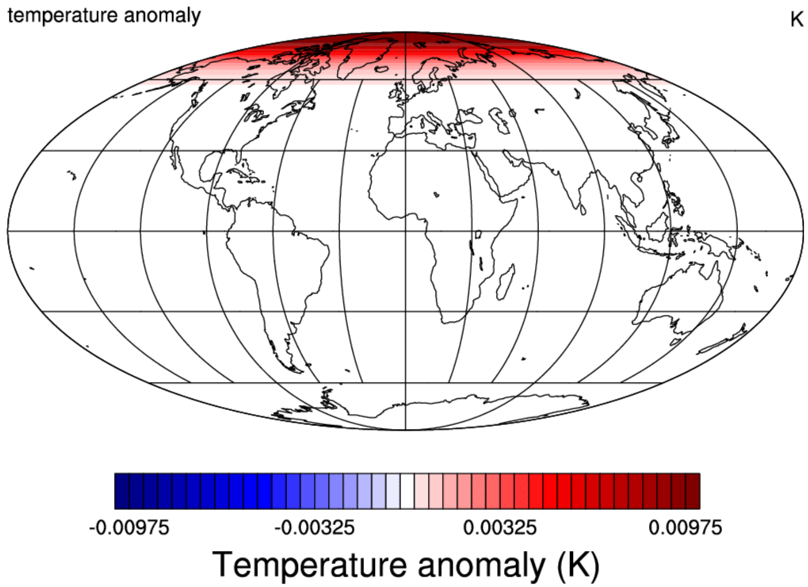
For problems with C-grid discretizations on non-quadrilateral grids see
Nickowicz, Gavrilov, Tosic (2002) MWR,
Thuburn, Ringler, Skamarock, Klemp (2009) JCP,
Gassmann (2011) JCP

(C)Additional advection by a solid body rotation velocity field $\boldsymbol{v}_0 = \boldsymbol{Q} \times \boldsymbol{r}$
→ test the coupling of fast (buoyancy, sound) and slow (advection) processes
Problem: solid body rotation field generates centrifugal forces!
Solution: $\boldsymbol{Q} = -\boldsymbol{\Omega}$ → similar to (A) in the absolute system

ICON (joint development of DWD/MPI-M) **simulation, f=0**

→ Talk by G. Zängl

in z=5 km



Small earth simulations

Wedi, Smolarkiewicz (2009) QJRMS

•$r_s = r_{earth} / 50 \sim 127$ km
simulations with $\Delta\varphi \sim 1° ... 0.0625°$ → $\Delta x \sim 2.2$ km ... 0.14 km
→ non-hydrostatic regime

•for runs *with* Coriolis force:
$f = f_{earth} \cdot 10 \sim 10^{-3}$ 1/s
→ dimensionless numbers

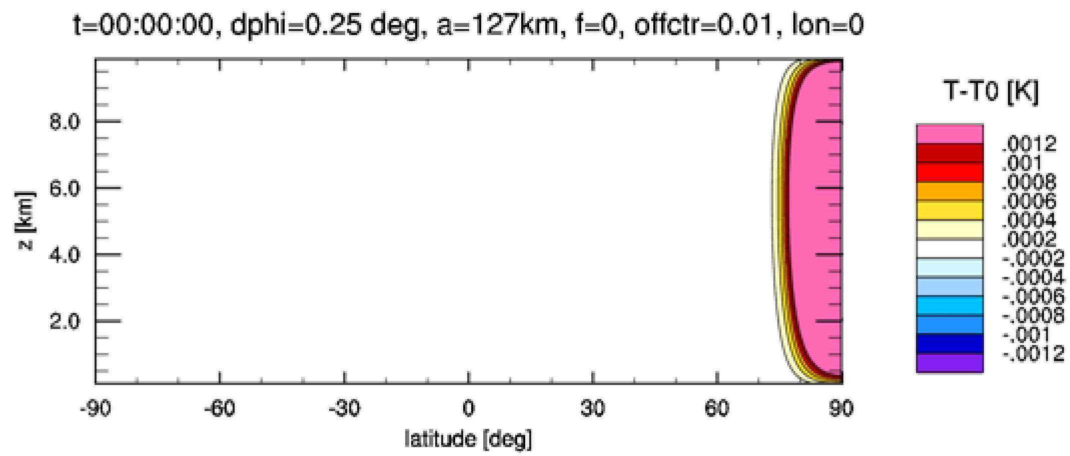
$$Ro = 0.2 \cdot Ro_{earth}$$

$$f / N = 10 \cdot f_{earth} / N \sim 0.05$$

Time evolution of T'

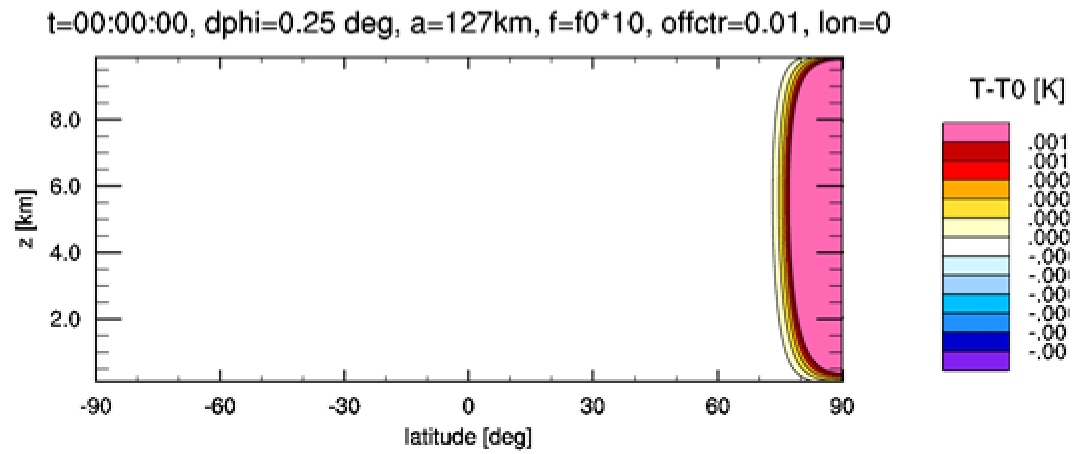
f=0

test scenario (A)



f≠0

test scenario (B)



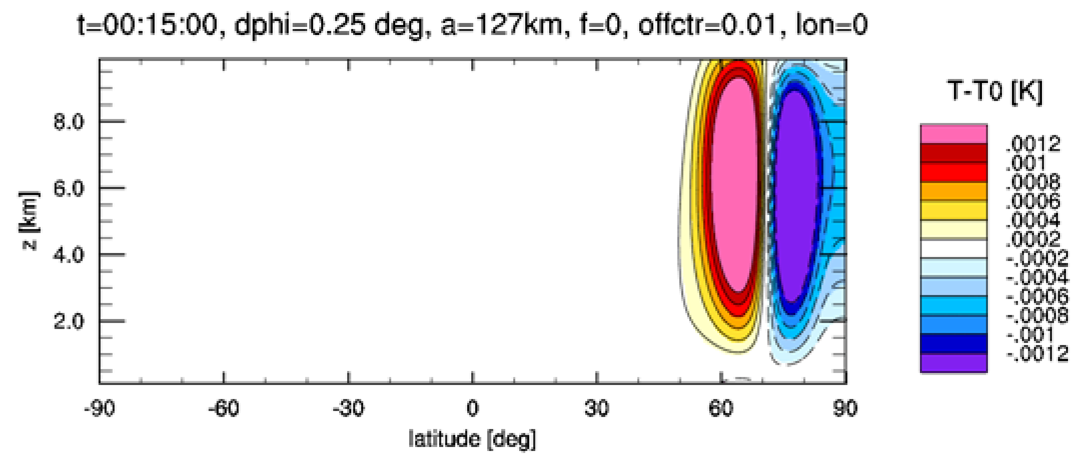
Black lines: analytic solution
Colours: ICON simulation

S Equ N

Time evolution of T'

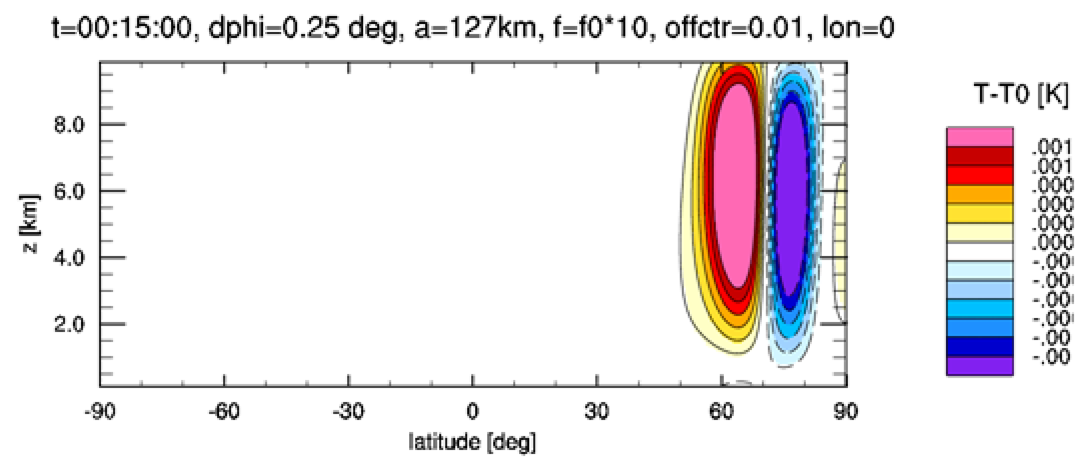
$f=0$

test scenario (A)



$f\neq0$

test scenario (B)



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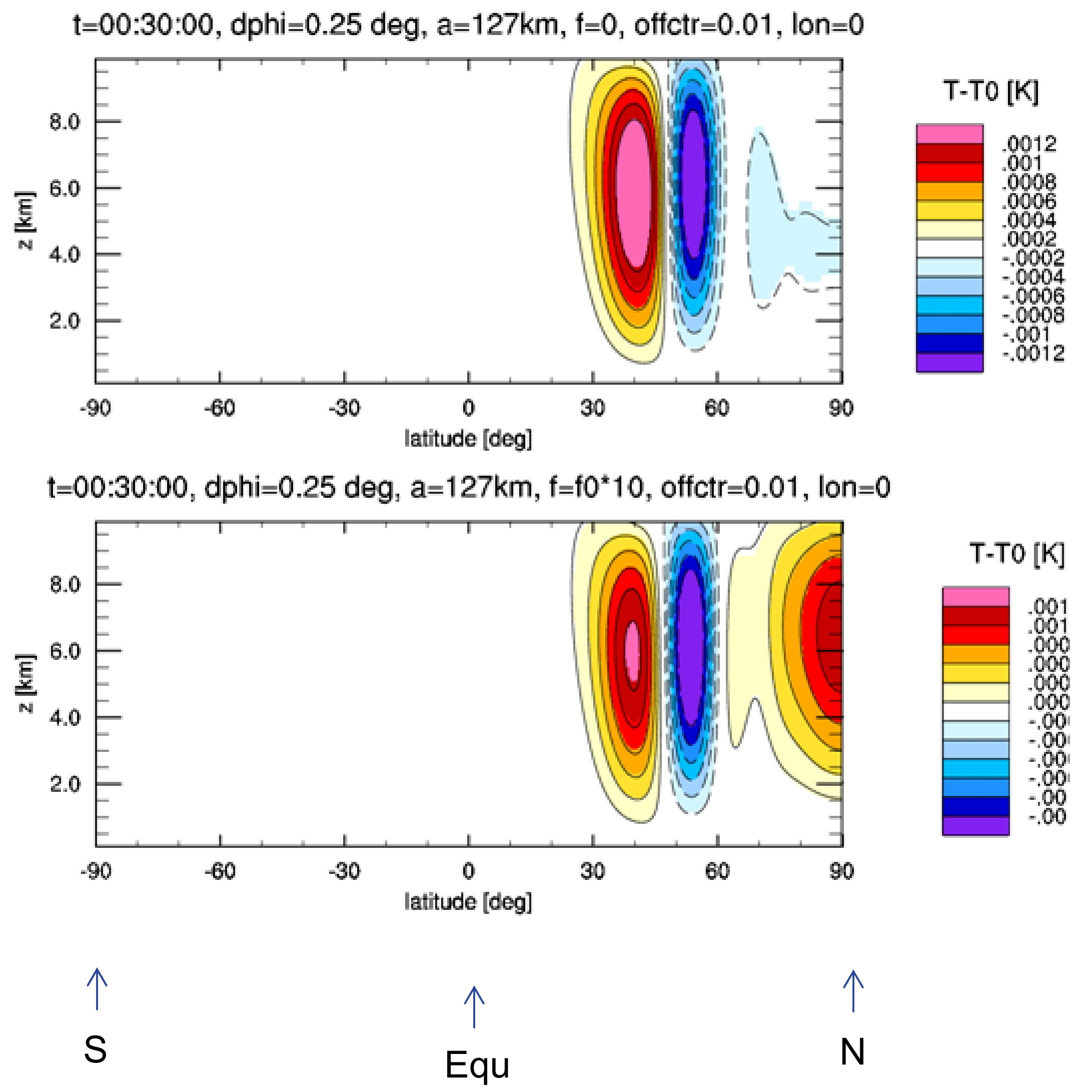
Time evolution of T‘

$f=0$

test scenario (A)

$f\neq 0$

test scenario (B)



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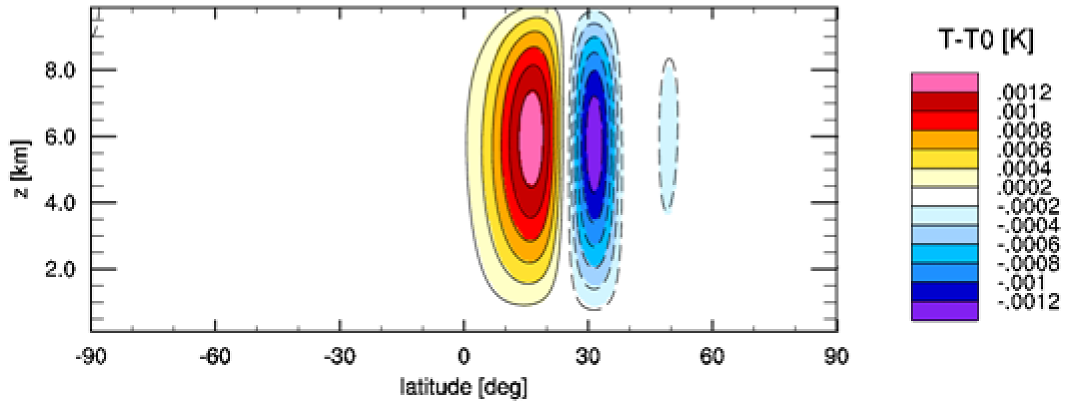
Time evolution of T‘

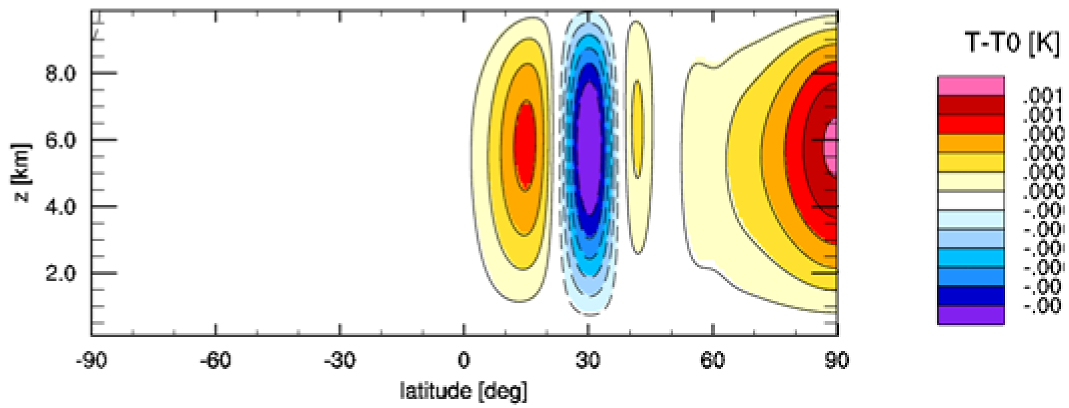
f=0

test scenario (A)



f≠0

test scenario (B)



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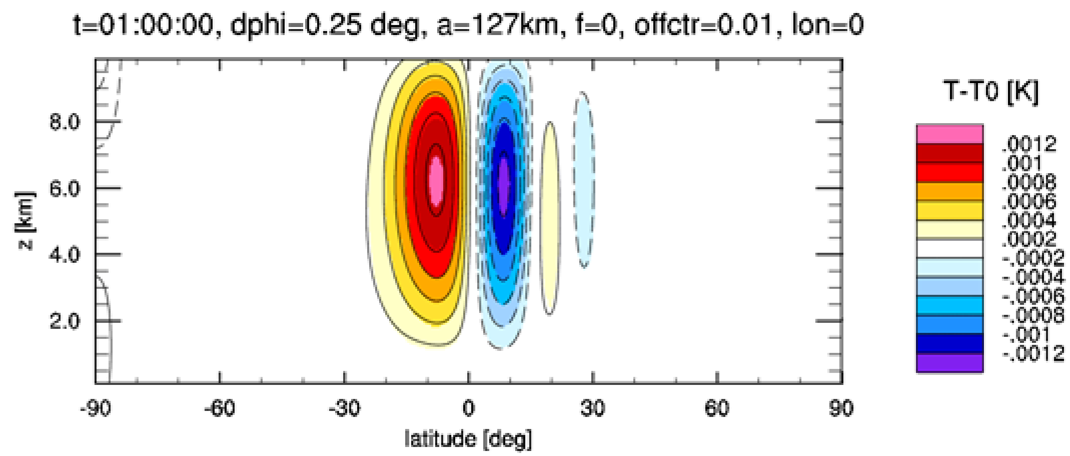
Time evolution of T‘

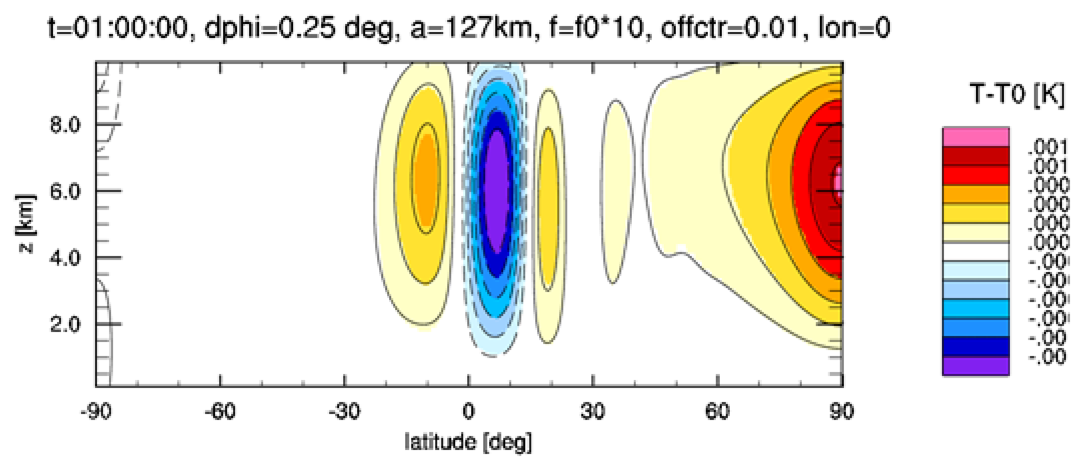
$f=0$

test scenario (A)



$f\neq0$

test scenario (B)



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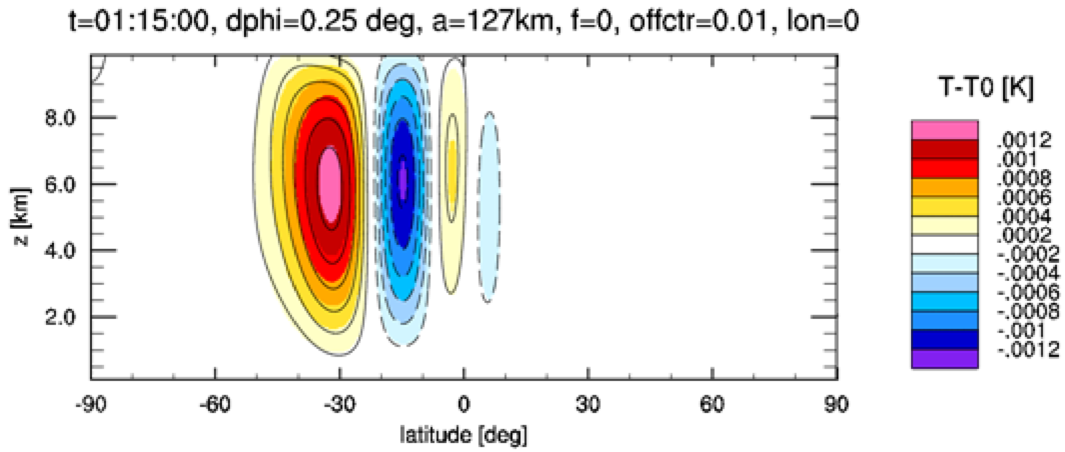
Time evolution of T‘

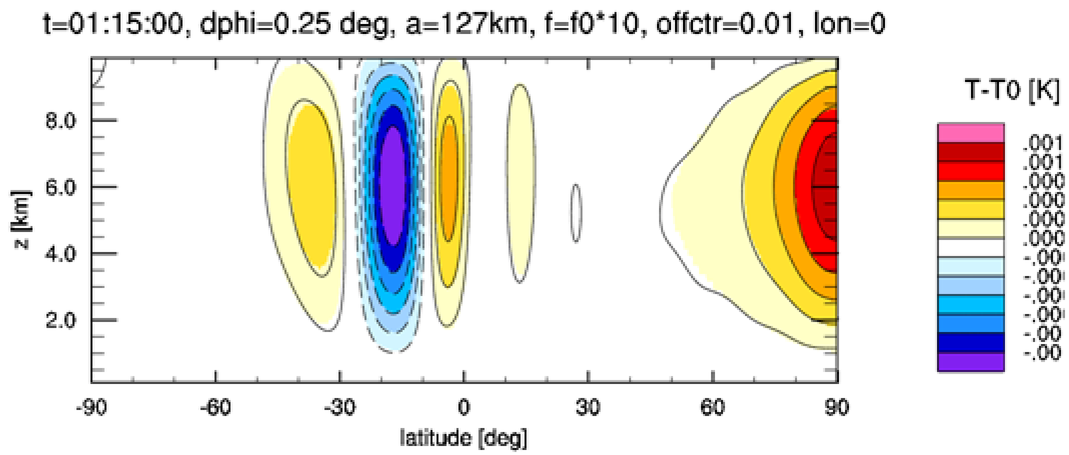
$f=0$

test scenario (A)



$f\neq 0$

test scenario (B)



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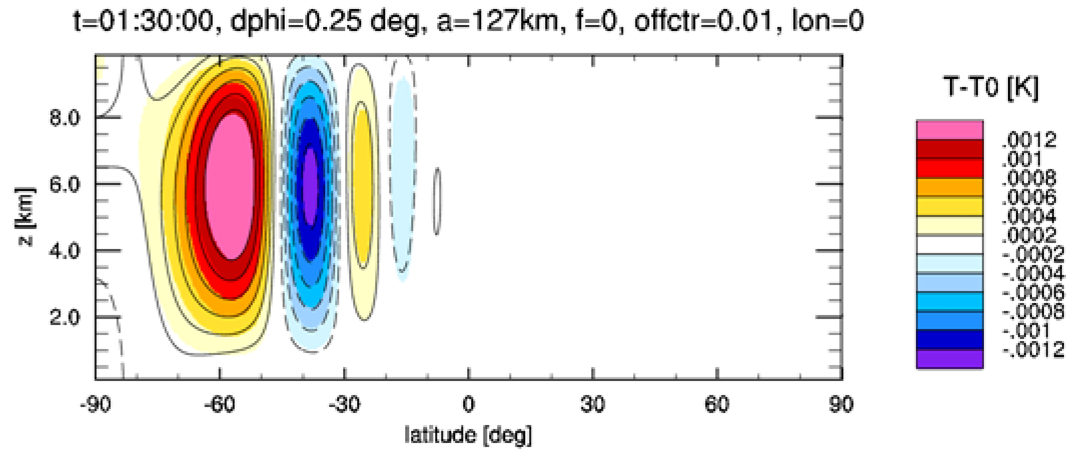
Time evolution of T‘

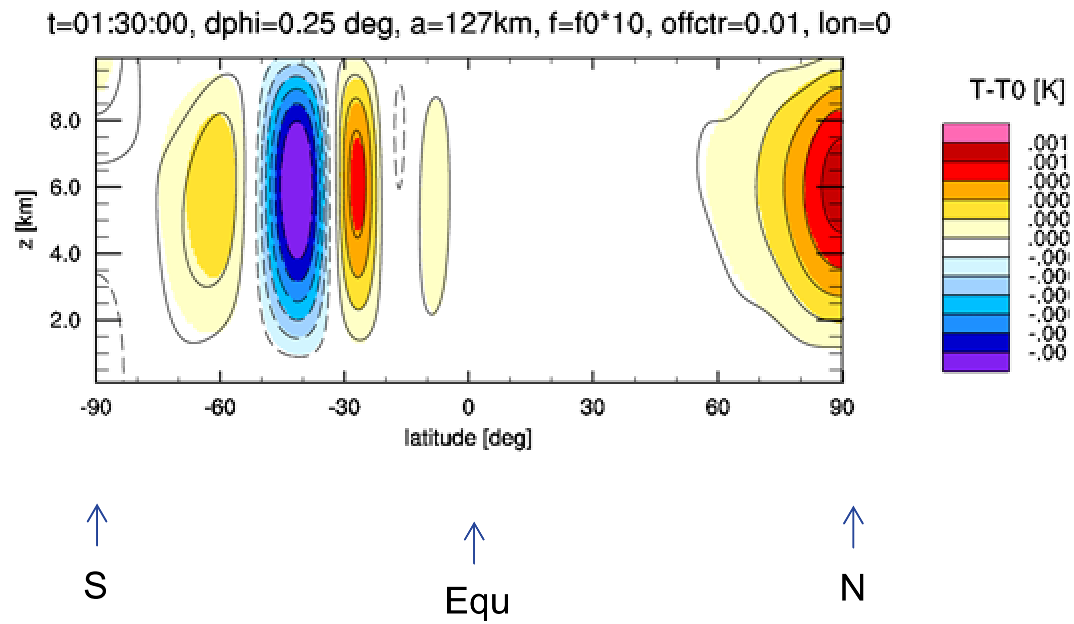
$f=0$

test scenario (A)



$f\neq0$

test scenario (B)

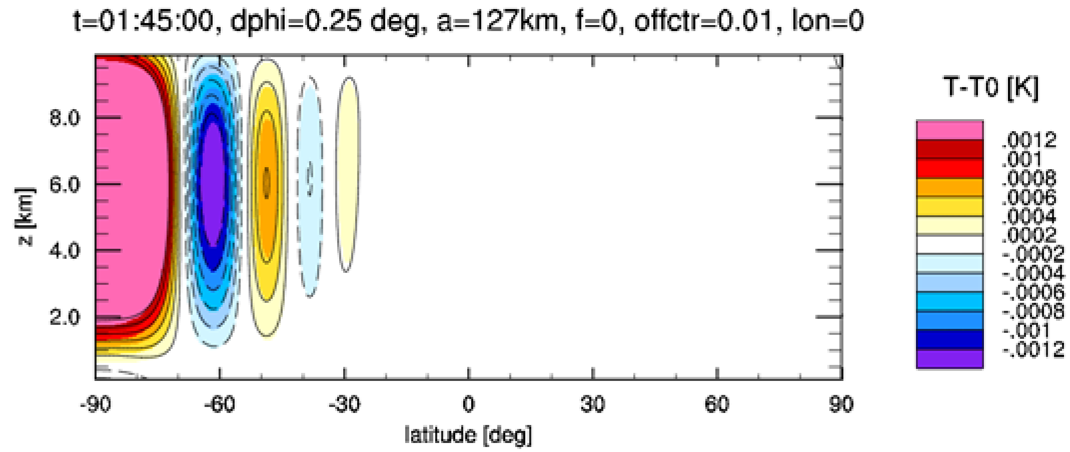


Black lines: analytic solution
Colours: ICON simulation

Time evolution of T'

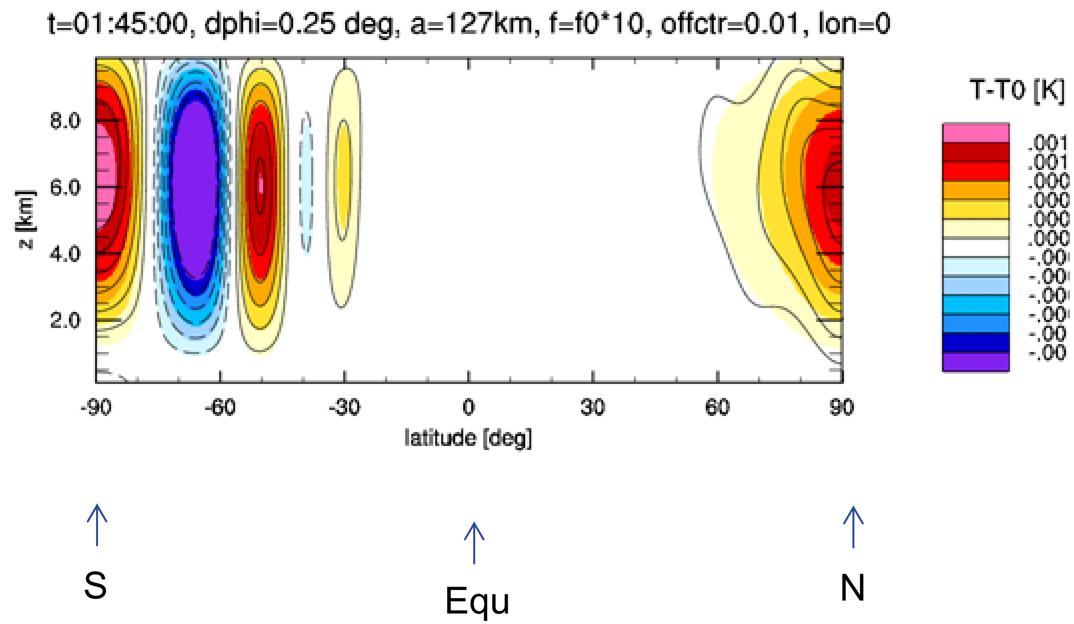
$f=0$

test scenario (A)



$f \neq 0$

test scenario (B)



Black lines: analytic solution
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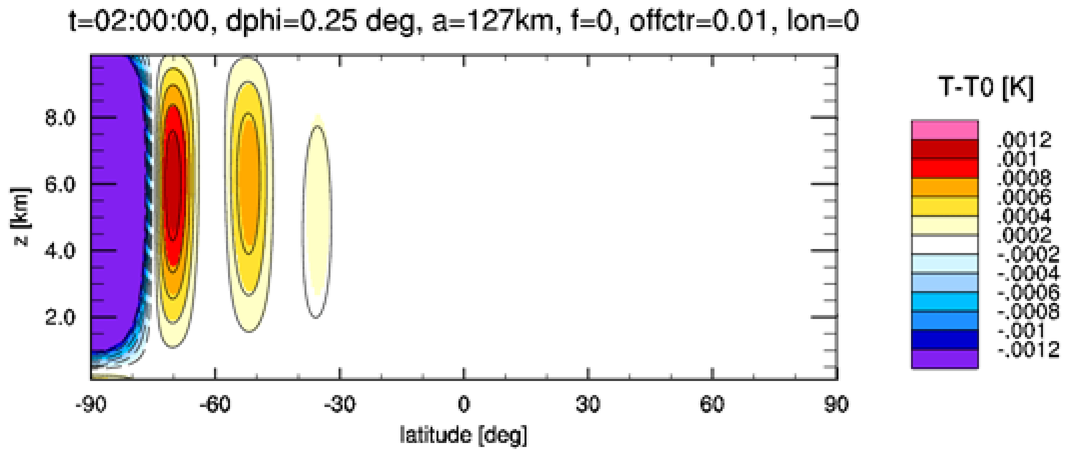
Time evolution of T‘

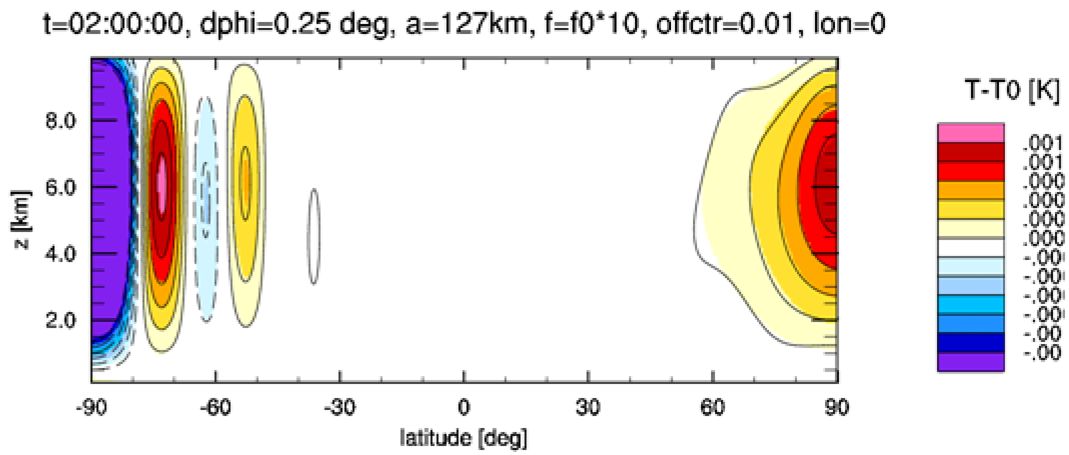
$f=0$

test scenario (A)



$f\neq0$

test scenario (B)



Black lines: analytic solution
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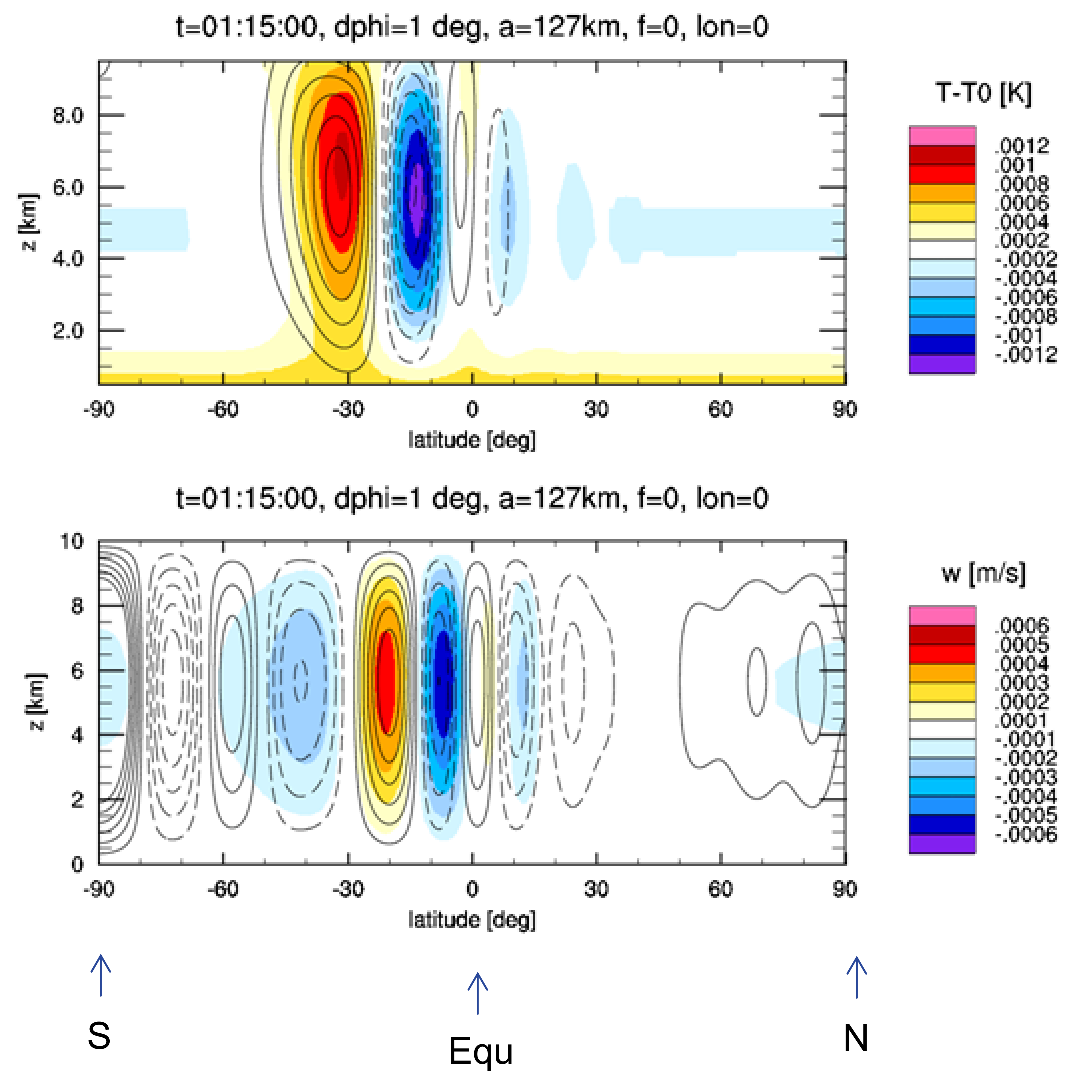
Convergence

R2B4-L10

$\Delta\phi \sim 1° \sim 2.2$ km
$\Delta z = 1000$ m

test scenario (A)



Black lines: analytic solution
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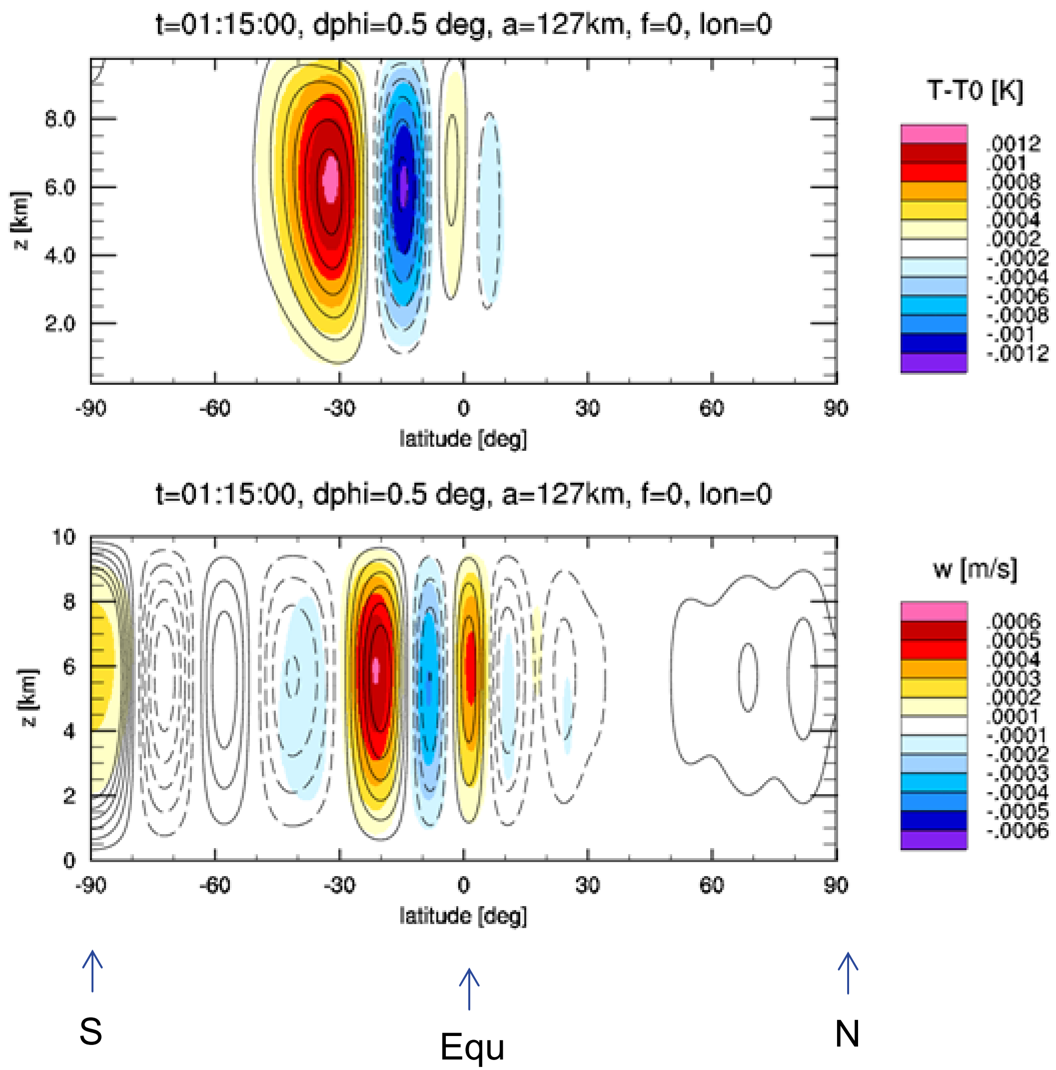
Convergence

R2B5-L20

$\Delta\phi \sim 0.5° \sim 1.1$ km
$\Delta z = 500$ m

test scenario (A)



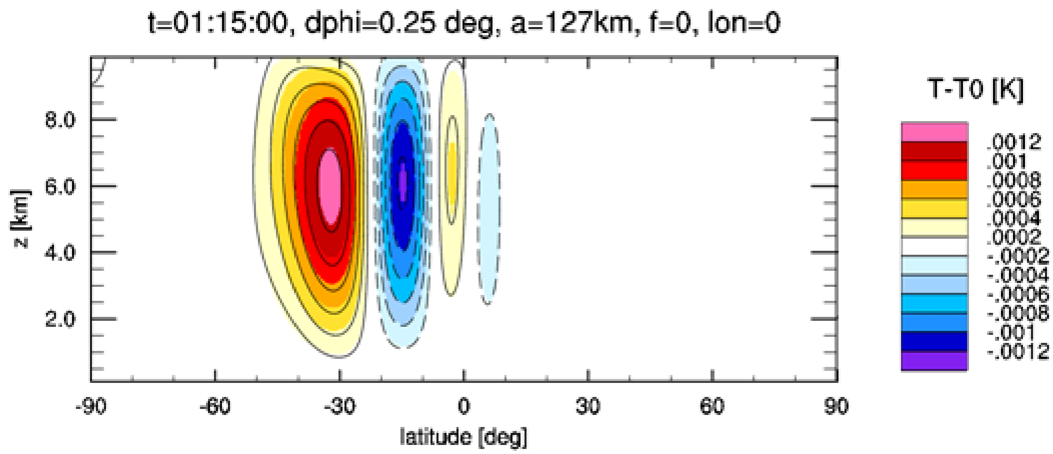
Black lines: analytic solution
Colours: ICON simulation

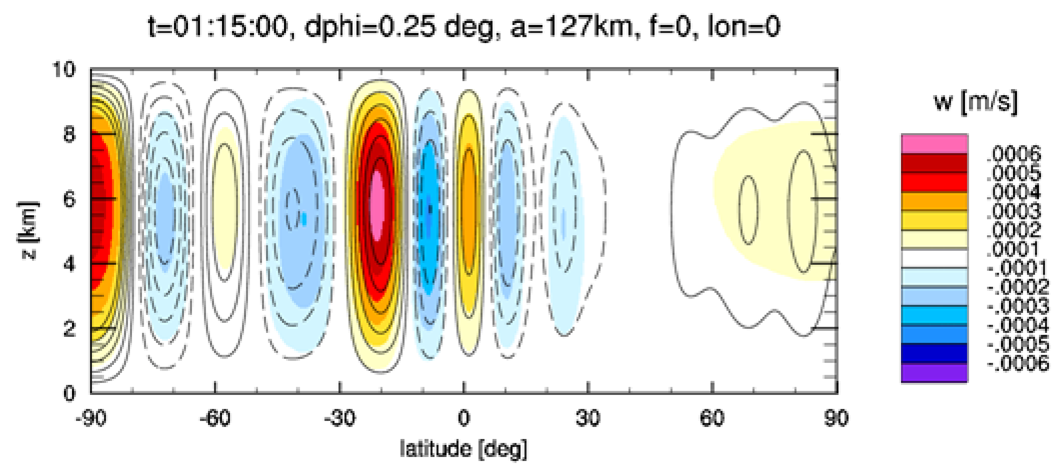
Convergence

R2B6-L40

$\Delta\phi \sim 0.25° \sim 0.55$ km
$\Delta z = 250$ m

test scenario (A)





S Equ N

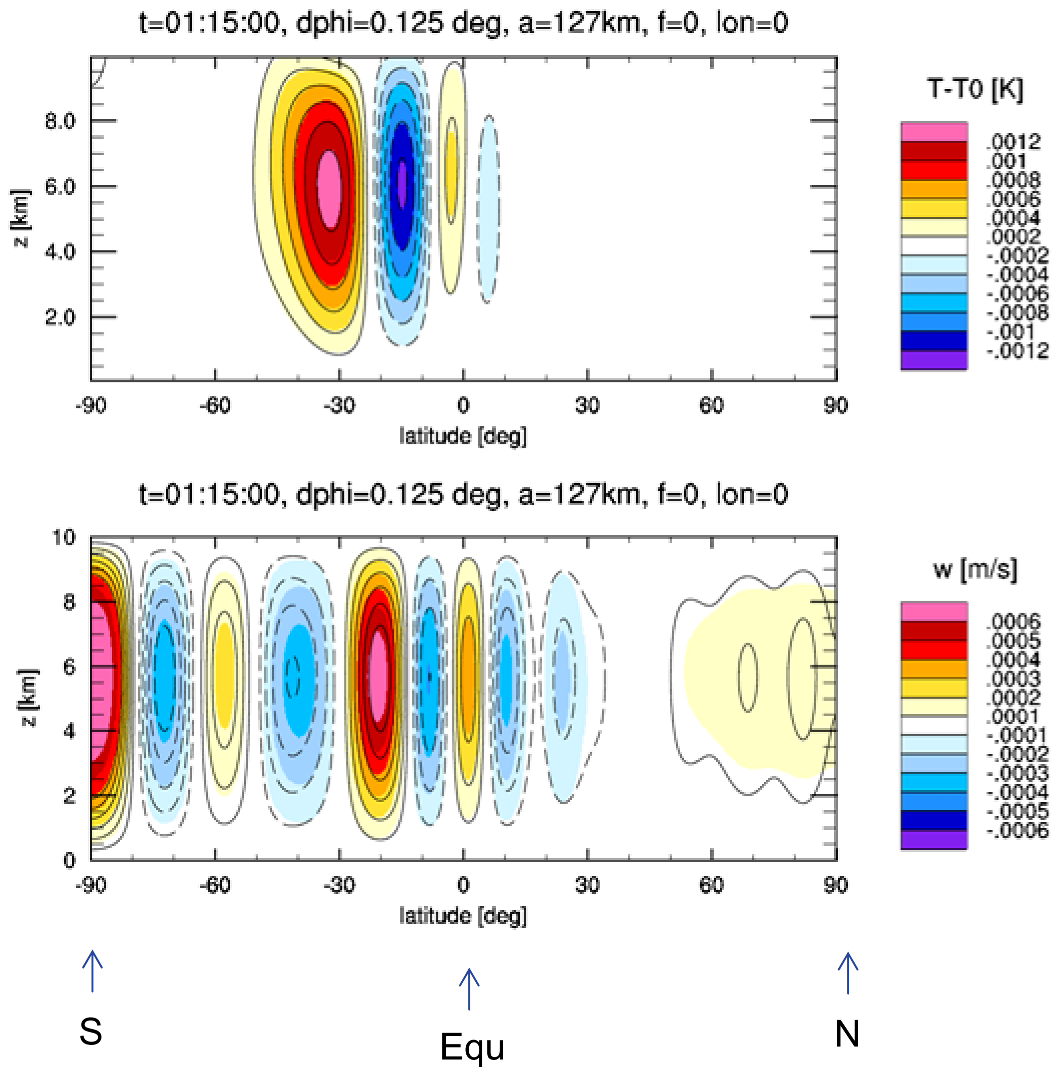
Black lines: analytic solution
Colours: ICON simulation

Convergence

R2B7-L80

$\Delta\phi \sim 0.125° \sim 0.28$ km
$\Delta z = 125$ m

test scenario (A)

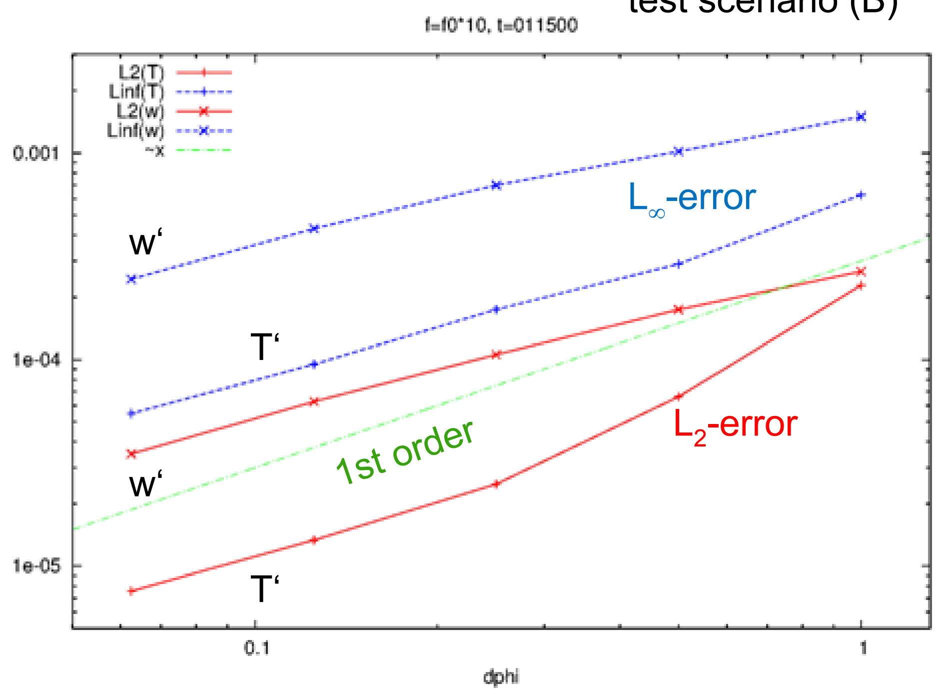


Black lines: analytic solution
Colours: ICON simulation

Convergence rate of the ICON model

- The ICON simulation with/without Coriolis force produces almost similar L_2, L_∞ errors
- Spatial-temporal convergence order of ICON is ~ 1

test scenario (B)



Test scenarios

(A) Only gravity wave and sound wave expansion

(B) …

(C) Additional advection by a solid body rotation velocity field $\boldsymbol{v}_0 = \boldsymbol{Q} \times \boldsymbol{r}$
→ test the coupling of fast (buoyancy, sound) and slow (advection) processes
Problem: solid body rotation field generates centrifugal forces!
Solution: $\boldsymbol{Q} = -\boldsymbol{\Omega}$ → similar to (A) in the absolute system
(analogous to *Läuter et al. (2005) JCP)*

Euler equations in spherical coordinates

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u - \frac{\tan\phi}{r} uv + \boxed{\frac{1}{r} uw} = - \frac{1}{\rho r \cos\phi} \frac{\partial p}{\partial \lambda} + 2\Omega(v \sin\phi \boxed{- w \cos\phi)}$$

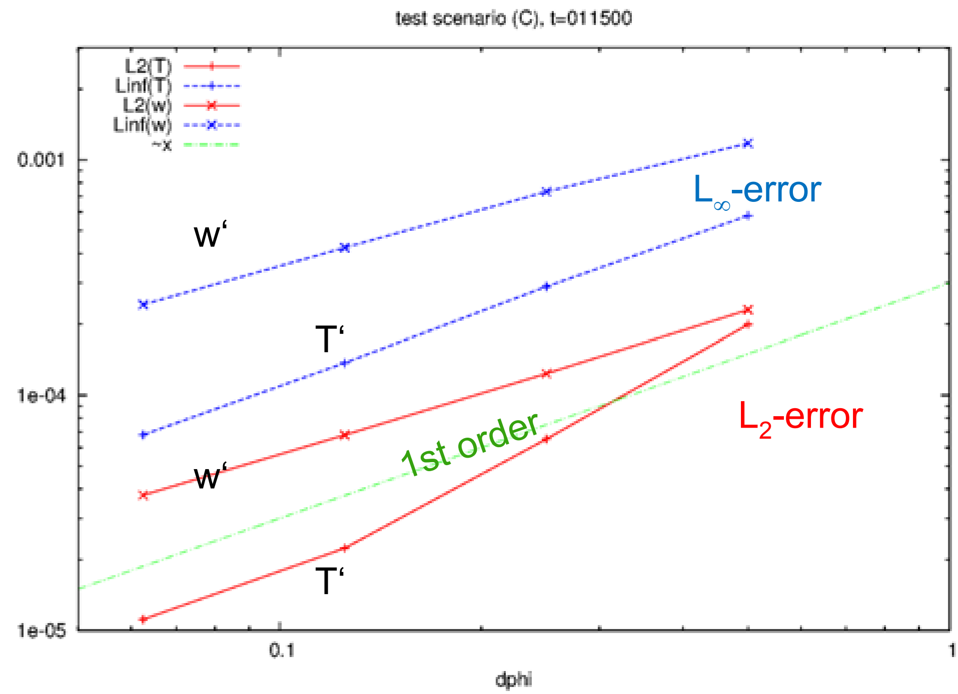
$$\frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla v + \frac{\tan\phi}{r} u^2 + \frac{1}{r} vw = - \frac{1}{\rho r} \frac{\partial p}{\partial \phi} - 2\Omega u \sin\phi - \Omega^2 r \cos\phi \sin\phi$$

$$\frac{\partial w}{\partial t} + \mathbf{v} \cdot \nabla w - \frac{u^2 + v^2}{r} = - \frac{1}{\rho} \frac{\partial p}{\partial r} - g + 2\Omega u \cos\phi + \Omega^2 r \cos^2\phi$$

Most deep terms are needed now for the analytic solution!
… but not all are contained in ICON

Test scenario (C) with the ICON model

Nevertheless:
The missing deep terms in the horizontal equations are not visible until 0.0625°: ICON converges ~ 1st order



Application of the analogous analytic solution for the expansion of waves on a 2D plane (*Baldauf, Brdar (2013) QJRMS)*

New fast waves development for the limited area model COSMO

Baldauf (2013) COSMO Tech. Rep. no. 21 (www.cosmo-model.org)

1. improvement of the vertical discretization:
use of weighted averaging operators for all vertical operations

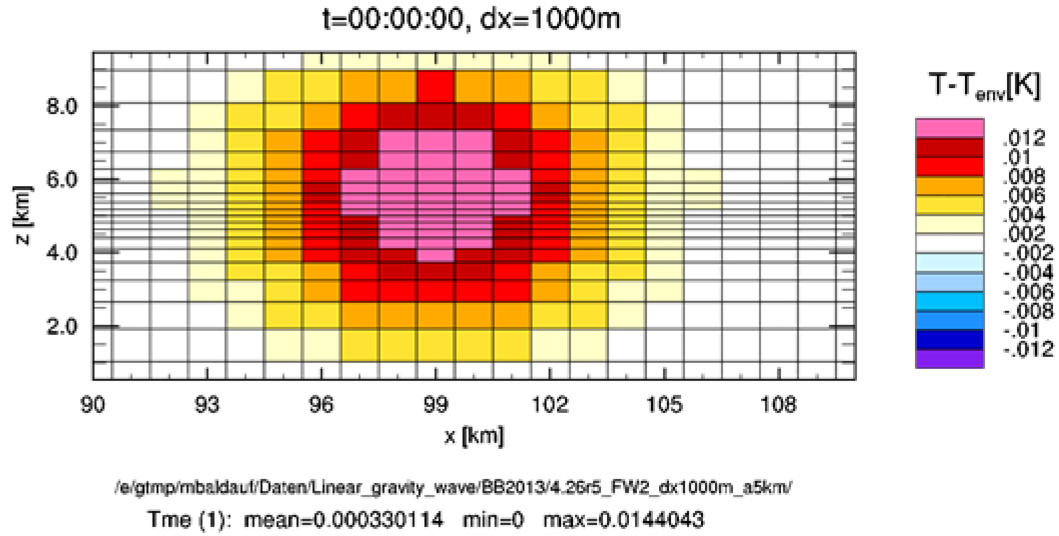
2. divergence in strong conservation form

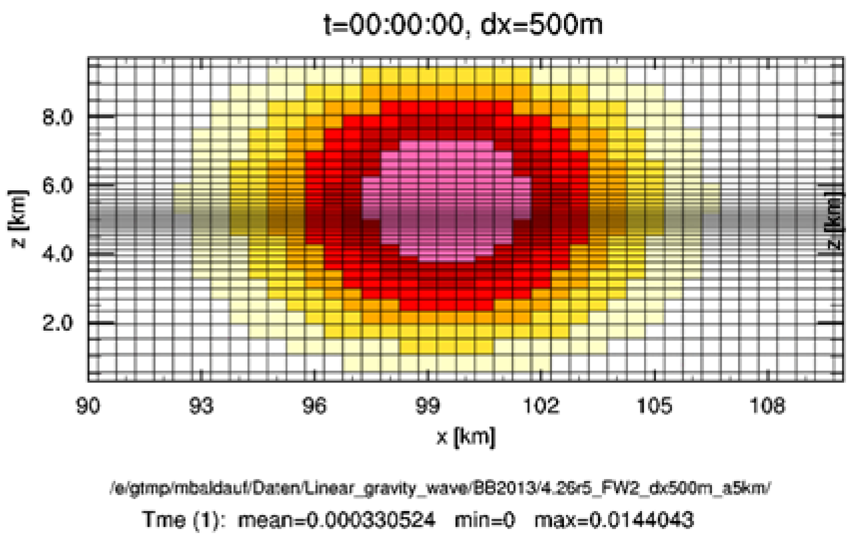
3. optional: complete 3D (=isotropic) divergence damping

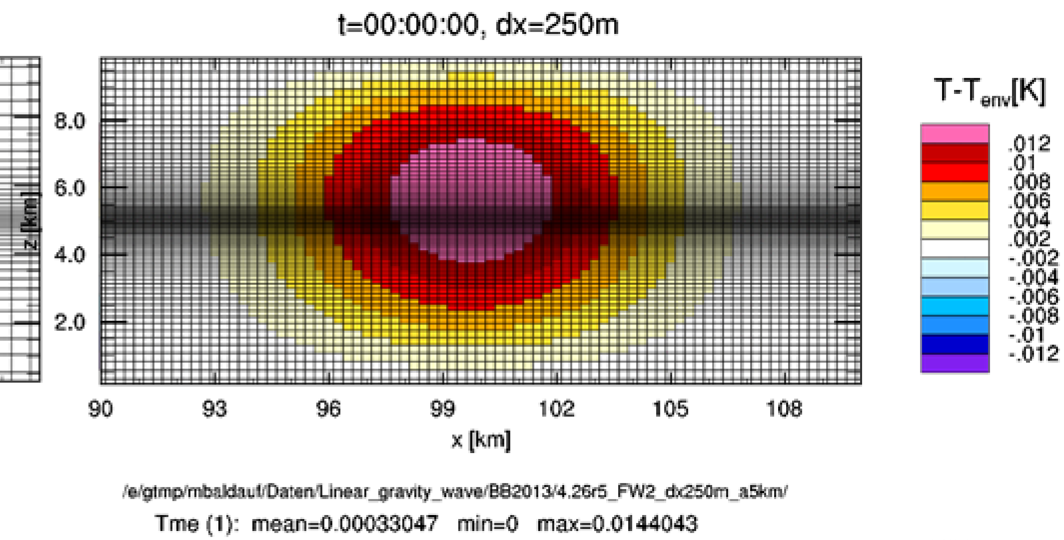
4. optional: Mahrer (1984) discretization of horizontal pressure gradients

Convergence test with vertically stretched grid

initial condition for T'
and grids
for the first 3 resolutions



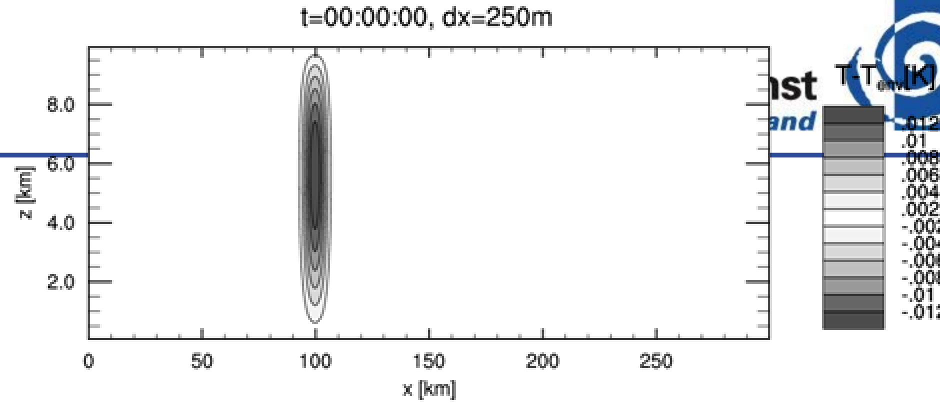




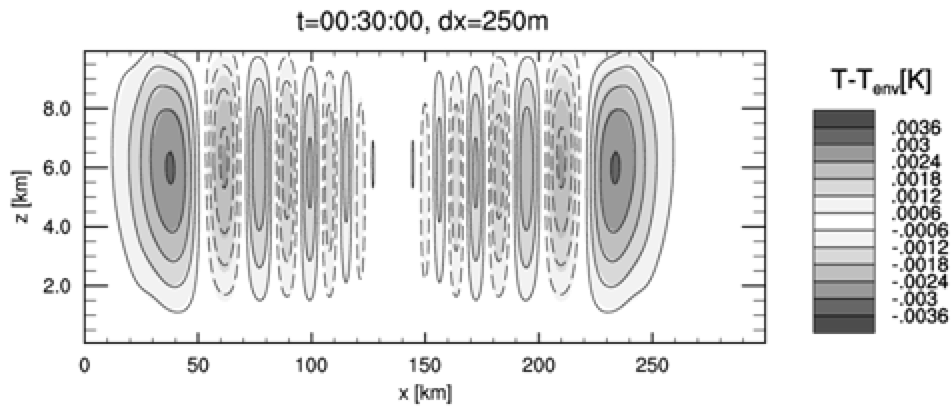
Initialization similar to
Skamarock, Klemp (1994)

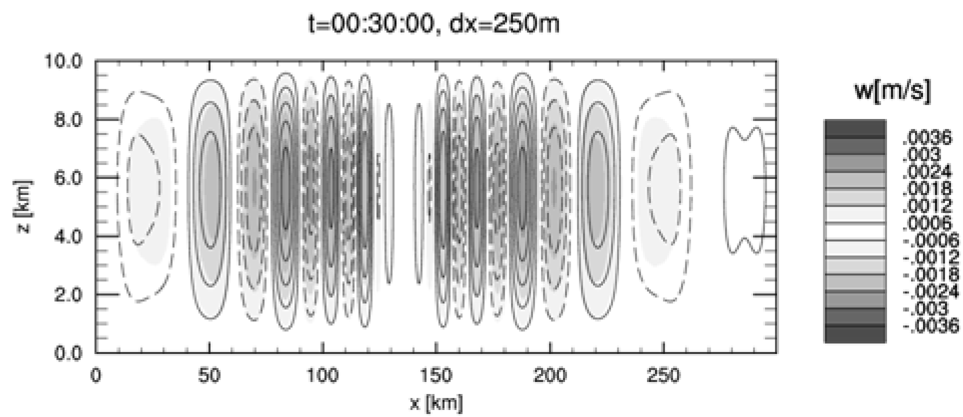
$$T'(x,z,t=0) = \Delta T \cdot e^{\frac{1}{2}\delta z} \cdot e^{-\frac{(x-x_c)^2}{d^2}} \cdot \sin \pi \frac{z}{H}$$

$$p'(x,z,t=0) = 0$$



Small scale test
with a basic flow U_0=20 m/s
f=0





Black lines: analytic solution
(Baldauf, Brdar (2013) QJRMS)

Shaded: COSMO

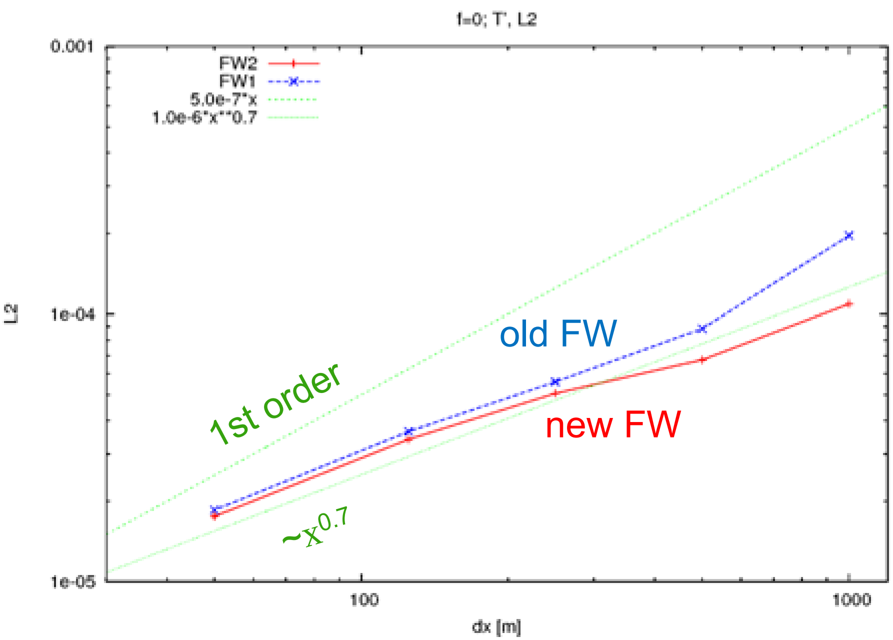
Convergence test with vertically stretched grid for old and new fast waves solver

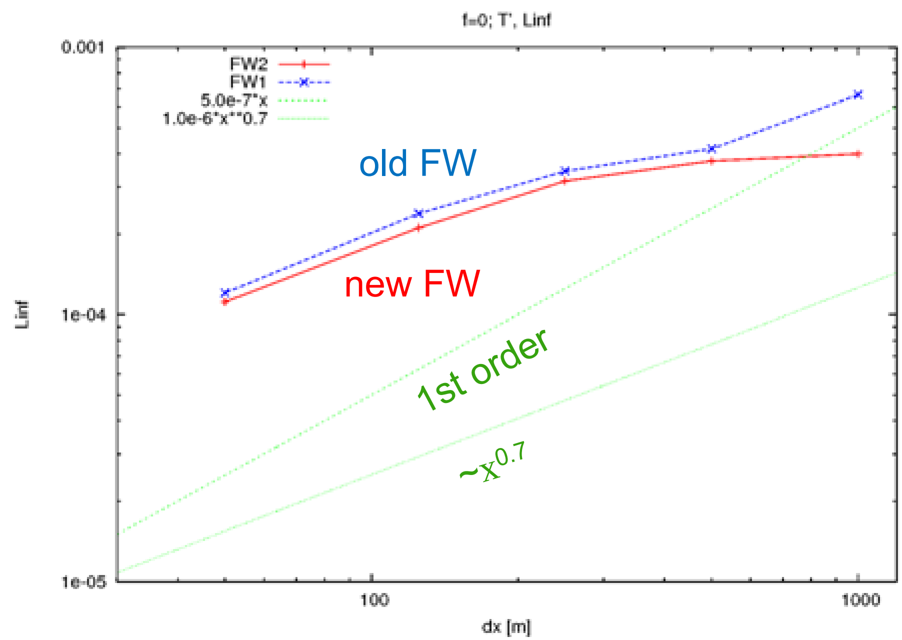
(against the analytic solution of *Baldauf, Brdar (2013) QJRMS)*

L_2-error

L_∞-error

T'





Summary

Deutscher Wetterdienst
Wetter und Klima aus einer Hand

•A new analytic solution of the compressible, non-hydrostatic Euler equations on the sphere and on a plane has been derived → a reliable solution for a well known test exists and can be used not only for qualitative comparisons but even as a reference solution for convergence tests

•The three proposed test scenarios exercise several important processes/terms and time integration schemes of the numerical model

•The test setup is also quite similar to one of the DCMIP 2012 test cases

•'standard' approximations used: shallow atmosphere, ‚global f-plane approx.' can be easily realised in every atmospheric model

•only one further approximation: linearisation (=controlled approx.)

•For fine enough resolutions ICON has a spatial-temporal convergence rate of about 1, no drawbacks visible

•Such tests can be used to evaluate improved discretizations (example: new fast waves solver in COSMO)

References:

Baldauf, Brdar (2013) QJRMS
Baldauf, Reinert, Zängl (2013) subm. to QJRMS