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Multilevel Time Integrators for Large–scale Atmospheric Flows

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Acknowledgements

Dahlem Research School

Berlin Mathematical School

Atmosphere dynamics: different spatial and temporal scales

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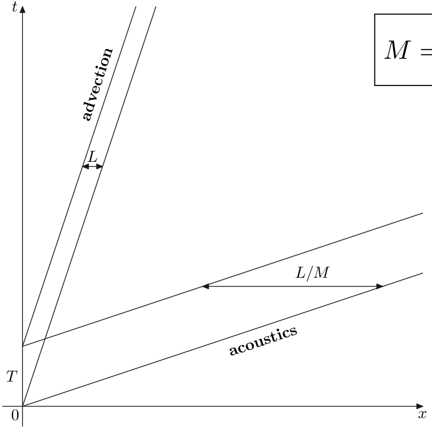

$$M = \frac{U}{c}$$

For atmospheric flows

$$0 < M \lesssim 0.3$$

Soundproof models

Sound waves: physically **unimportant**, computationally **demanding**

For **explicit** methods, **CFL**

$$\frac{c\Delta t}{\Delta x} < 1$$

Analytical approach: **reduced** models

Anelastic-like models on small scales, **hydrostatic** on large scales

Sound waves

Numerical approach: semi-implicit fully compressible models

The implicit part handles sound and/or gravity modes

No stability constraints, but dispersive

Target

A conservative, semi–implicit, fully compressible model

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A conservative, semi–implicit, fully compressible model

- **Second–order accurate throughout**
- **Consistent with soundproof models**
- **With selective control over resolved scales/effects**

Compressible Euler Equations

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = -\rho g \mathbf{k}$$

$$\alpha P_t + \nabla \cdot (P \mathbf{v}) = 0$$

Equation of state, ideal gas:

$$P = \rho\theta = \frac{p_{ref}}{R} \left(\frac{p}{p_{ref}} \right)^{1/\gamma}$$

Initial and boundary conditions

Durran's pseudo–incompressible model: $\alpha = 0, \; P = \overline{P}(z)$

Numerics

Fractional step method (Klein (2009)):

- **Predictor**: auxiliary **hyperbolic** system
- **Corrector**: two **projections** for **non–compliant** fluxes and **use** of p^n

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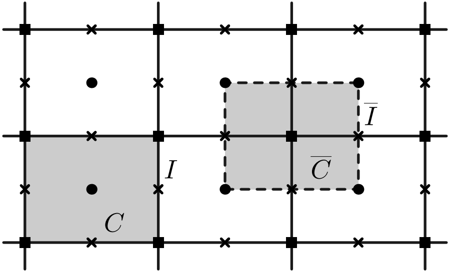

Predictor

Finite volume scheme in conservation form

Time integration: second-order SSP Runge-Kutta, $\mathbf{CFL_{adv}} < 1$

$$(\rho\mathbf{v})_C^{n+1,*} = (\rho\mathbf{v})_C^n - \frac{\Delta t}{|C|}\left(\int_{\partial C} \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})\,\mathrm{d}l\right)^{n+\frac{1}{2}} - \frac{\Delta t}{|C|}\left(\int_{\partial C} p\cdot\mathbf{n}\,\mathrm{d}l\right)^{n}$$

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- **Velocities not compliant with:**

$$\left[\alpha P_t + \nabla \cdot (P \mathbf{v}) \right]_{n+\frac{1}{2}} = 0$$

- **Old pressure used in the momentum equation**

Corrector step – First projection

Integrate ρv over a half time step, replace in

$$[\alpha P_t + \nabla \cdot (P\mathbf{v})]_{n+\frac{1}{2}} = 0$$

with

$$\frac{\partial P}{\partial t} \approx \frac{\partial P}{\partial p} \frac{\delta p}{\Delta t}$$

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Helmholtz equation

$$\boldsymbol{\alpha} C_1 \delta p + \nabla \cdot (C_2 \nabla \delta p) = \nabla \cdot \left[(P\mathbf{v})^{n+\frac{1}{2}} \right]$$

Equivalent to wave equation for p, consistent with soundproof limit

$$C_1 \propto \frac{M^2}{CFL_{adv}^2}$$

Corrector step – Second projection

On the dual grid, integrate $\rho \mathrm{v}$ over a full time step, replace in

$$\frac{P^{n+1} - P^n}{\Delta t} + \frac{1}{2} \nabla \cdot \left[(P\mathbf{v})^{(n+1)} + (P\mathbf{v})^{(n)} \right] = 0 \qquad \text{TRAP}$$

or

$$\frac{\frac{3}{2} P^{n+1} - 2P^n + \frac{1}{2} P^{n-1}}{\Delta t} + \nabla \cdot (P\mathbf{v})^{(n+1)} = 0 \qquad \text{BDF2}$$

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TRAP

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$$\frac{\frac{3}{2} P^{n+1} - 2P^n + \frac{1}{2} P^{n-1}}{\Delta t} + \nabla \cdot (P\mathbf{v})^{(n+1)} = 0$$

BDF2

Second Helmholtz equation

$$\boldsymbol{\alpha} C_1 \delta p_n + \nabla \cdot (C_2 \nabla \delta p_n) = RHS$$

Momentum and pressure corrected, full second–order scheme

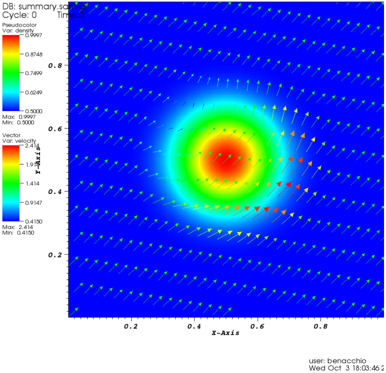
Implementation

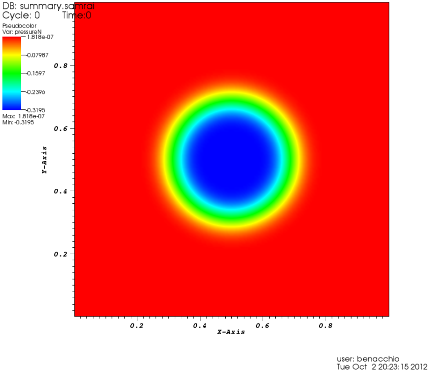
C++ code, SAMRAI & Hypre

Overhead of compressibility: added diagonal term

Traveling rotating vortex in $[0,1]^2$ m, Kadioglu et al. (2008)

Exact solution in the incompressible case, $(\mathbf{u_{bg}}, \mathbf{v_{bg}}) = (1, 1)$

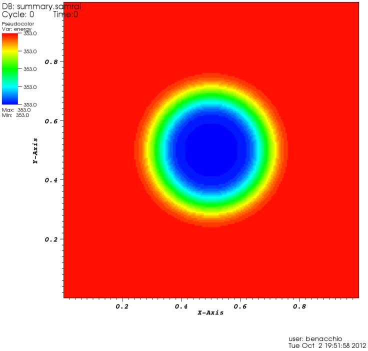




$$\rho_{\mathbf{ref}} = 0.5\,\mathrm{kg/m^3},\ \mathbf{p_{ref}} = 101625\,\mathrm{Pa},\ \mathbf{T_{ref}} = 706.098\,\mathrm{K}$$

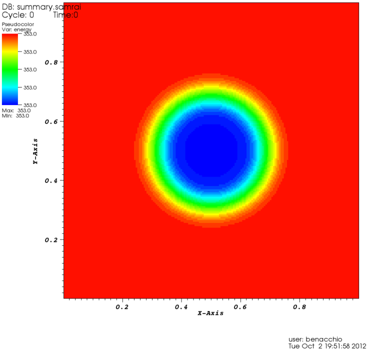
$$\mathbf{M} \approx 4.5\text{e-}03$$

$P_{t=0}$ **derived from** $p_{t=0}$ **through EOS**



Results

$P_{t=0}$ derived from $p_{t=0}$ through EOS



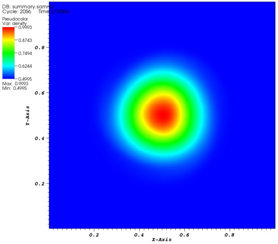
192×192 **cells**

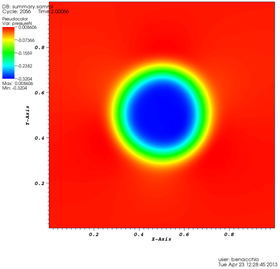
$$\Delta t = CFL_{adv} \Delta x / U_{max}$$

$$CFL_{adv} = \boxed{\mathbf{0.45}}$$

$$CFL_{ac} \approx \boxed{\mathbf{99.3}}$$

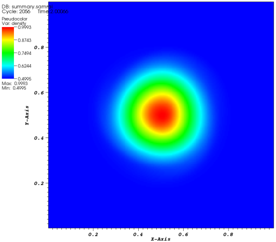
COMPRESSIBLE

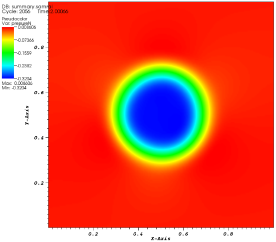




$T = 2\,s$

COMPRESSIBLE





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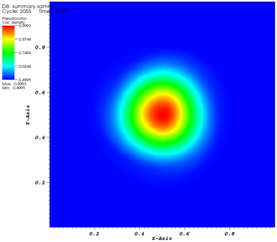

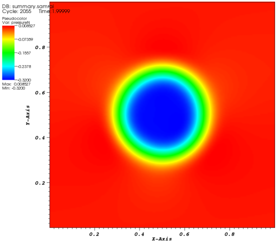

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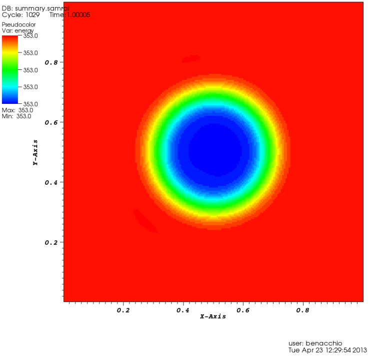
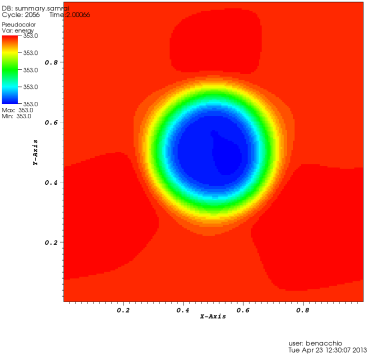
Transport of P

$T = 1\,s$ $T = 2\,s$





$$\alpha P_t + \nabla \cdot (P\mathbf{v}) = 0$$

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Simple acoustic wave

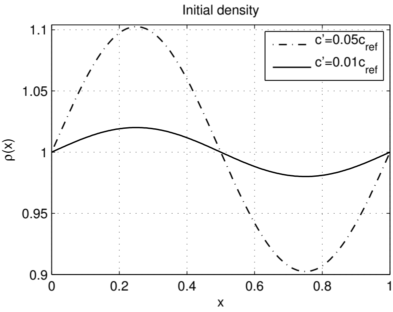
$$u = \frac{2}{\gamma - 1} c' \sin\left(\frac{2\pi}{L} x\right)$$

$$\rho = \rho_{ref} \left[1 + \frac{c'}{c_{ref}} \sin\left(\frac{2\pi}{L} x\right)\right]^{\frac{2}{\gamma - 1}}$$

$$p = p_{ref} \left[1 + \frac{c'}{c_{ref}} \sin\left(\frac{2\pi}{L} x\right)\right]^{\frac{2\gamma}{\gamma - 1}}$$

Rieper(2011), Vater(2013)

$\mathbf{x} \in [0, 1]$, **periodic BCs**



$$\rho_{\mathbf{ref}} = 1\,\mathbf{kg/m^3}, \gamma = 2$$

$$\mathbf{T_{ref}} \approx 353\,\mathbf{K},\ \mathbf{c_{ref}} \approx 450\,\mathbf{m/s}$$

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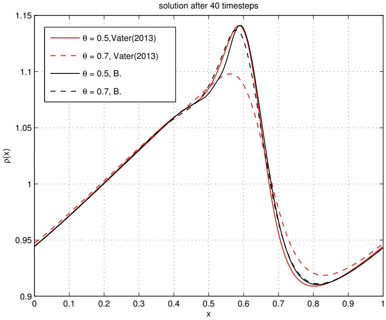
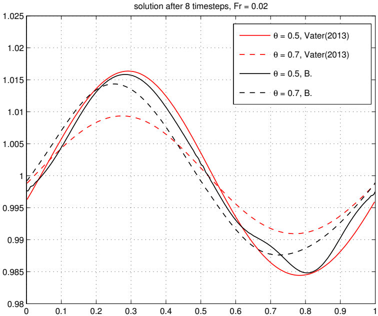
$$\Delta x = 1/256\,\mathrm{m},\ \mathrm{CFL_{adv}} = 0.77$$





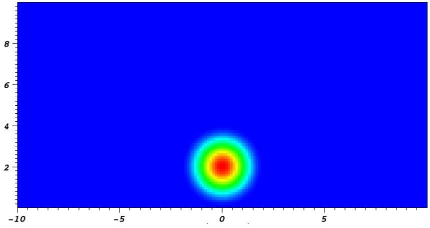
$c' = 0.05 c_{ref},\ M(t=0) \approx 0.1$

$c' = 0.01 c_{ref},\ M(t=0) \approx 0.02$

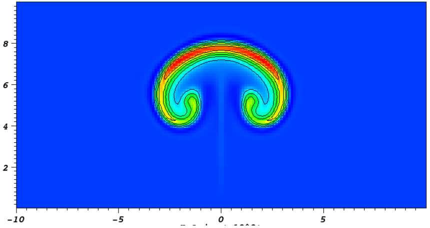
Warm air bubble in $[-10, 10] \times [0, 10]\,\mathbf{km}$ neutrally stratified atmosphere, Klein(2009)

$$\theta'(x,z) = \begin{cases} 2\,\mathrm{K}\cos^2(\frac{\pi}{2}r) & (r \leq 1) \\ 0 & \text{otherwise} \end{cases}$$

BDF2, $\mathbf{CFL_{adv} = 0.5}$, $\Delta x = 125\,\mathbf{m}$



$\mathbf{T = 0\,s}$



$\mathbf{T = 1000\,s}$, $\delta\theta'_{\mathbf{max}} \approx \mathbf{1.7\,K}$

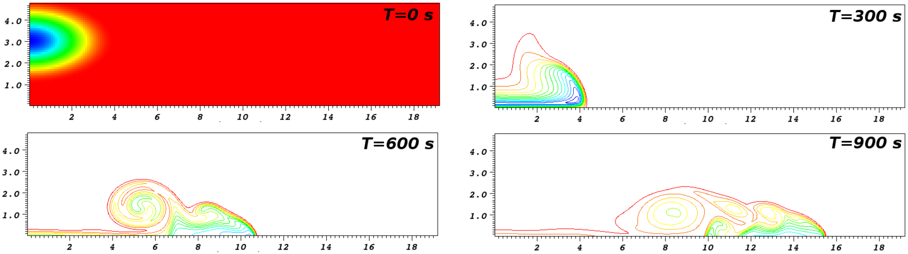
Density **current**

Straka et al. '93, Mueller et al. '12

BDF2, $\mathbf{CFL_{adv}} = \mathbf{0.65}$, $\Delta x \approx 28\,\mathbf{m}$

$$T' = \begin{cases} -15\,\mathrm{K}\left[\frac{1}{2}\left(1 + \cos\left(\frac{\pi}{2} r\right)\right)\right] & (r \leq 1) \\ 0 & \text{otherwise} \end{cases}$$

$$+\ \mu = 75.0\,\mathbf{m}^2/\mathbf{s}$$



Front position: $x = \underline{15497.71}\,$**m (M. '12:** $\underline{15452.4}\,$**m)**

$|\theta'_{\mathbf{max}}| = \underline{10.13}\,$**K (M. '12:** $\underline{9.8}\,$**K)**

Summary

Fully compressible, semi–implicit model for the simulation of multiscale atmospheric processes

Summary

Fully compressible, **semi–implicit** model for the simulation of **multiscale** atmospheric processes

- **Compatible** with **pseudo–incompressible** soundproof model
- **Conservative** and **second–order** accurate
- **Platform** to investigate processes **interactions** at large scales

Vater (2013): multigrid in time, blend of implicit schemes

Damping and dispersion properties used to eliminate unresolved modes and transport long waves accurately

Multiscale features of large–scale test cases modelled with controlled treatment of acoustic/gravity waves

Corrector step – First projection

Integrating $\rho\mathbf{v}$ over a half time step

$$(\rho\mathbf{v})^{n+\frac{1}{2}} = (\rho\mathbf{v})^{n} - \frac{\Delta t}{2}\left[\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})\right]^{n+\frac{1}{4}} - \frac{\Delta t}{2}\nabla\left(p^{n} + \delta p\right)$$

$$= (\rho\mathbf{v})^{n+\frac{1}{2},*} - \frac{\Delta t}{2}\nabla\delta p$$

Corrector step – First projection

Integrating $\rho \mathbf{v}$ over a half time step

$$
\begin{aligned}
(\rho \mathbf{v})^{n+\frac{1}{2}} &= (\rho \mathbf{v})^{n} - \frac{\Delta t}{2} \left[\nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) \right]^{n+\frac{1}{4}} - \frac{\Delta t}{2} \nabla \left(p^{n} + \delta p \right) \\
&= (\rho \mathbf{v})^{n+\frac{1}{2},*} - \frac{\Delta t}{2} \nabla \delta p
\end{aligned}
$$

With

$$
\frac{\partial P}{\partial t} = \frac{\partial P}{\partial p} \frac{\partial p}{\partial t} \approx \frac{R^{-\gamma}}{\gamma} \left(\frac{P}{p_{ref}} \right)^{1-\gamma} \frac{\delta p}{\Delta t}
$$

Corrector step – First projection

Integrating $\rho\mathbf{v}$ over a half time step

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With

$$\frac{\partial P}{\partial t} = \frac{\partial P}{\partial p}\frac{\partial p}{\partial t} \approx \frac{R^{-\gamma}}{\gamma}\left(\frac{P}{p_{ref}}\right)^{1-\gamma}\frac{\delta p}{\Delta t}$$

the constraint

$$\left[\alpha P_t + \nabla\cdot(P\mathbf{v})\right]_{n+\frac{1}{2}} = 0$$

Corrector step – First projection

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With

$$\frac{\partial P}{\partial t} = \frac{\partial P}{\partial p}\frac{\partial p}{\partial t} \approx \frac{R^{-\gamma}}{\gamma}\left(\frac{P}{p_{ref}}\right)^{1-\gamma}\frac{\delta p}{\Delta t}$$

the constraint

$$\left[\alpha P_t + \nabla\cdot(P\mathbf{v})\right]_{n+\frac{1}{2}} = 0$$

yields a Helmholtz equation:

$$-\frac{\alpha}{\gamma\Delta t}\left[P^{n+\frac{1}{2},*}\right]^{1-\gamma} R^{-\gamma} p_{ref}^{\gamma-1}\delta p + \nabla\cdot\left[\frac{\Delta t}{2}\left(\frac{P}{\rho}\right)^{n+\frac{1}{2},*}\nabla\delta p\right] = \nabla\cdot\left[(P\mathbf{v})^{n+\frac{1}{2},*}\right]$$

Corrector step – First projection

If $\delta p = p^{n+1} - p^n$, equivalent formulation with $\alpha = 1$

$$-\frac{2}{\Delta t^2 c^2}\left(p^{n+1} - p^n\right) + \nabla^2 p^{n+1} = \frac{2}{\Delta t}\rho^{n+\frac{1}{2},*}\nabla \cdot \mathbf{v}^n$$

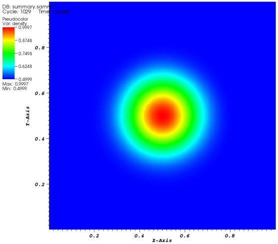
- **If** $CFL_{adv} = \dfrac{U\Delta t}{\Delta x} \quad \Longrightarrow \quad \dfrac{1}{\Delta t^2 c^2} = \dfrac{M^2}{CFL_{adv}^2 \Delta x^2}$
- **A finite centred difference discretization of the hyperbolic pressure equation**

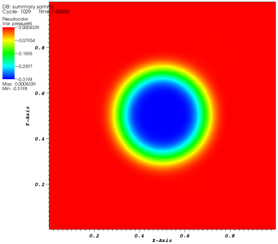
$$-\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \nabla^2 p = 0$$

gives (in progress...)

$$-\frac{2}{c^2 \Delta t^2}\left(p^{n+1} - p^n\right) + \frac{1}{4}\nabla^2 p^{n+1} = \frac{2}{\Delta t}\,\rho^n\,\nabla \cdot \mathbf{v}^*$$

COMPRESSIBLE





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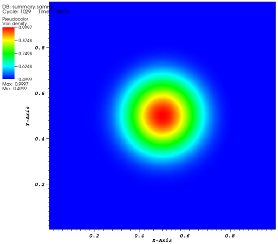
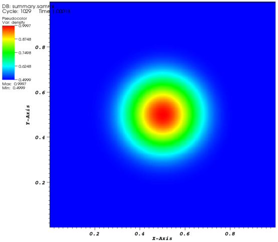
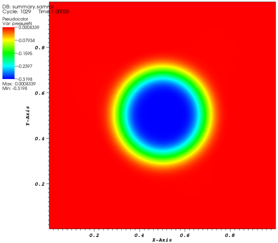
$T = 1\,s$

COMPRESSIBLE

PSEUDO-INCOMPRESSIBLE







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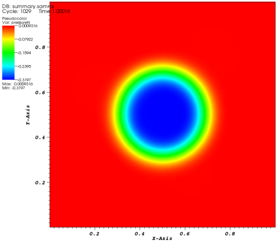

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