# Representation of topography by the thin-wall approximation in a height coordinate

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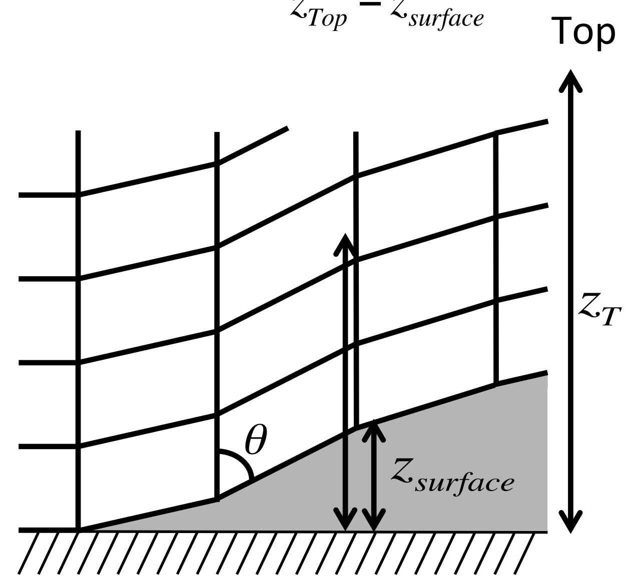
# Contents

# 1. Background

- ✓ In higher resolution models (≤ 10km) , the detailed topographical data is required.
- ✓ In terrain-following approach, only the upper and lower boundary conditions are imposed.
- ✓ This approach shows a relativity decent performance over gentler slopes.
- ✓ Over steeper slopes, this approach induces large truncation errors and tends to be numerically unstable.

Basic terrain-following

$$z^* = z_{Top} \frac{z - z_{surface}}{z_{Top} - z_{surface}}$$



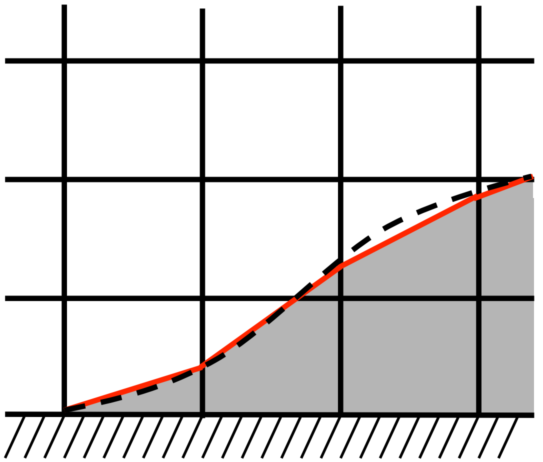

The terrain-following coordinate representation of a topography

➡ We focus on the alternatives to the terrain-following approach.

# Other representations

✓ **Cut cell method**

- This method satisfies a conservation law using a finite-volume method.
- Small cell problem occurs.



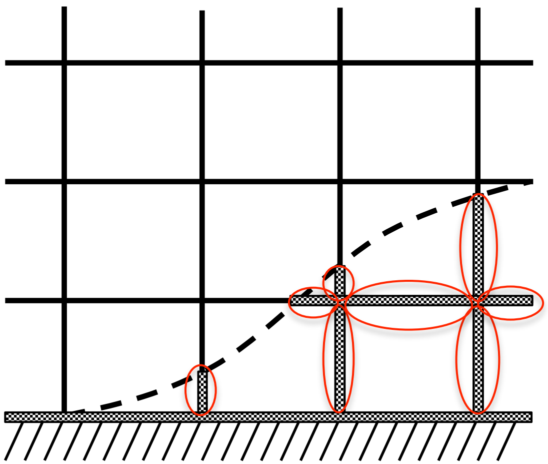
Cut cell method

✓ **Thin-wall approximation**

- The assumption that volume of cells intersected by the surface remains a full atmospheric cell.
- The parts under topography is represented by thin walls.
- Advection-form equations are used in pervious works such as

  Steppeler et al. (2002, 2006, 2011),

  Lock (2008), Lock et al. (2012).



Thin-wall approximation

# Purposes

- ✓ To implement the representation of topography by the thin-wall approximation to fully compressible flux-form equations.
- ✓ To confirm the conservation of momentum flux.

| | Steppeler et al. (2002) | Our approach |
|---|---|---|
| Equations | Advection-form<br>$$\frac{\partial T}{\partial t}=-u\frac{\partial T}{\partial x}-w\frac{\partial T}{\partial z}-\frac{p}{\rho C_v}\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)$$<br>$$\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x}-w\frac{\partial u}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$<br>$$\frac{\partial w}{\partial t}=-u\frac{\partial w}{\partial x}-w\frac{\partial w}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}-g$$<br>$$\frac{\partial p}{\partial t}=-u\frac{\partial p'}{\partial x}-w\frac{\partial p'}{\partial z}+g\rho_0 w-\frac{C_p}{C_v}p\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)$$ | Flux-form<br>$$\frac{\partial \rho'}{\partial t}=-\frac{\partial \rho u}{\partial x}-\frac{\partial \rho w}{\partial z}$$<br>$$\frac{\partial \rho u}{\partial t}=-\frac{\partial \rho uu}{\partial x}-\frac{\partial \rho wu}{\partial z}-\frac{\partial p'}{\partial x}$$<br>$$\frac{\partial \rho w}{\partial t}=-\frac{\partial \rho uw}{\partial x}-\frac{\partial \rho ww}{\partial z}-\frac{\partial p'}{\partial z}-\rho' g$$<br>$$\frac{\partial \rho e}{\partial t}=-\frac{\partial \rho uh}{\partial x}-\frac{\partial \rho wh}{\partial z}+u\frac{\partial p'}{\partial x}+w\frac{\partial p'}{\partial z}+w\frac{\partial p_s}{\partial z}$$ |
| Boundary conditions | Upwind difference | Velocities is set to zero at sides of a scalar cell. |

# 2. Model description

- **Equations（Satoh 2002）**

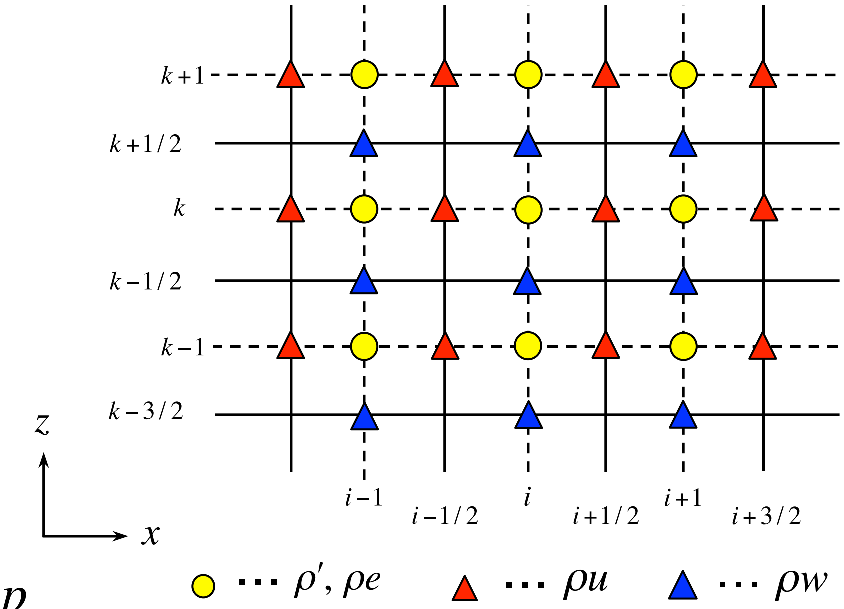
$$\frac{\partial \rho'}{\partial t} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial z}$$

$$\frac{\partial \rho u}{\partial t} = -\frac{\partial \rho uu}{\partial x} - \frac{\partial \rho wu}{\partial z} - \frac{\partial p'}{\partial x}$$

$$\frac{\partial \rho w}{\partial t} = -\frac{\partial \rho uw}{\partial x} - \frac{\partial \rho ww}{\partial z} - \frac{\partial p'}{\partial z} - \rho' g$$

$$\frac{\partial \rho e}{\partial t} = -\frac{\partial \rho uh}{\partial x} - \frac{\partial \rho wh}{\partial z} + u\frac{\partial p'}{\partial x} + w\frac{\partial p'}{\partial z} + w\frac{\partial p_s}{\partial z}$$

Here, $h = e + \frac{p}{\rho}, \quad p' = p - p_s, \quad \rho' = \rho - \rho_s$



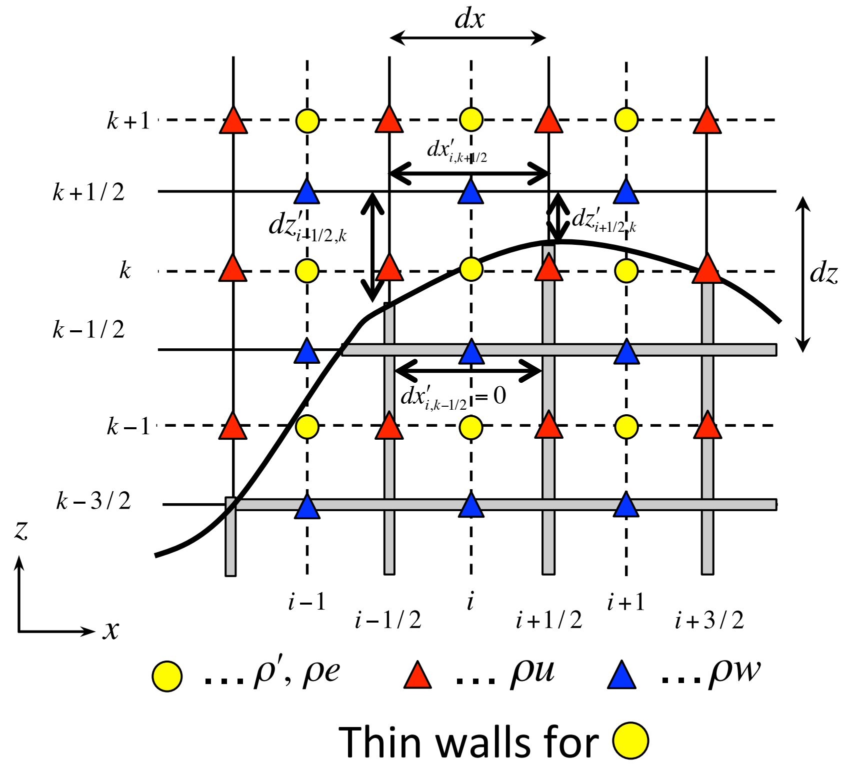

- **Discretization**

| | |
|---|---|
| Time integration | RK3（Williamson 1980) |
| Spatial difference | Second order central difference scheme |
| Grid system | C-grid / Lorentz-grid |

# 3. Thin-wall approximation

$$F_z \equiv \frac{\text{[ dz': the part of grid that appears above the surface]}}{\text{[ dz : total vertical grid interval ]}}$$

$$F_x \equiv \frac{\text{[ dx': the part of grid that appears above the surface]}}{\text{[ dx : total horizontal grid interval ]}}$$




$$\left(\frac{\partial \rho'}{\partial t}\right)_{i,k} = -\delta_x\left(\underline{F_z}\rho u\right) - \delta_z\left(\underline{F_x}\rho w\right)$$

$$\left(\frac{\partial \rho u}{\partial t}\right)_{i-1/2,k} = -\delta_x\left(\underline{F_z}\,\overline{\rho u}^x\,\overline{u}^x\right)_{i-1/2,k} - \delta_z\left(\underline{F_x}\,\overline{\rho w}^x\,\overline{u}^z\right)_{i-1/2,k} - \delta_x\left(p'\right)_{i-1/2,k}$$

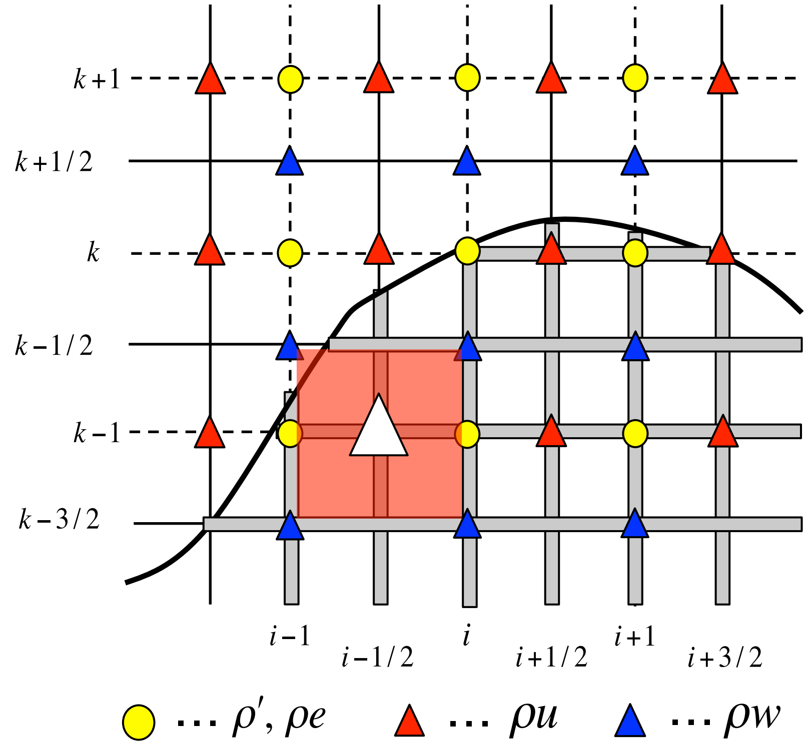
$$\left(\frac{\partial \rho w}{\partial t}\right)_{i,k-1/2} = -\delta_x\left(\underline{F_z}\,\overline{\rho u}^z\,\overline{w}^x\right)_{i,k-1/2} - \delta_z\left(\underline{F_x}\,\overline{\rho w}^z\,\overline{w}^z\right)_{i,k-1/2} - \delta_z\left(p'\right)_{i,k-1/2} - \overline{\rho'}^z_{i,k-1/2}\,g$$

$$\left(\frac{\partial \rho e}{\partial t}\right)_{i,k} = -\delta_z\left(\underline{F_z}\rho u\overline{h}^x\right)_{i,k} - \delta_z\left(\underline{F_x}\rho w\overline{h}^z\right)_{i,k} + \overline{u}^x_{i,k}\delta_x\left(\overline{p'}^x\right)_{i,k} + \overline{w}^z_{i,k}\delta_z\left(\overline{p'}^z\right)_{i,k} + \overline{w}^z_{i,k}\delta_z\left(\overline{p_s}^z\right)_{i,k}$$

# Boundary conditions

✓ Thin walls are defined at side of a cell for each variable.

→ It is necessary to handle all thin walls depending on a location relative to the surface.
The following conditions are imposed.




$$u_{i-1/2,k} = 0 \quad if \quad (F_z)_{i-1/2,k} = \frac{(dz')_{i-1/2,k}}{dz} = 0$$

$$w_{i,k-1/2} = 0 \quad if \quad (F_x)_{i,k-1/2} = \frac{(dx')_{i,k-1/2}}{dx} = 0$$

The velocity is set to zero if the side face of a cell defined scalar quantity is consists of thin wall only.

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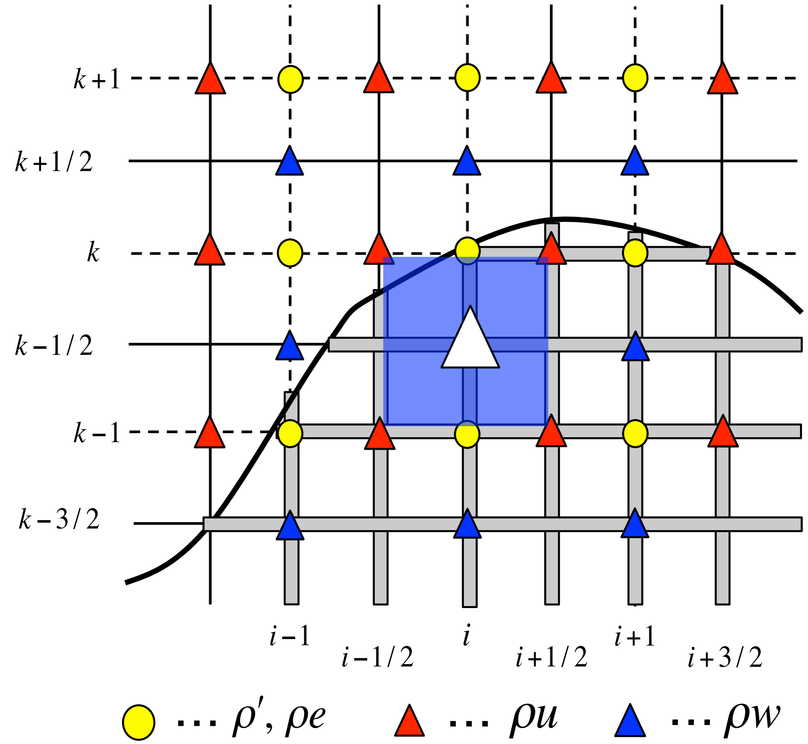


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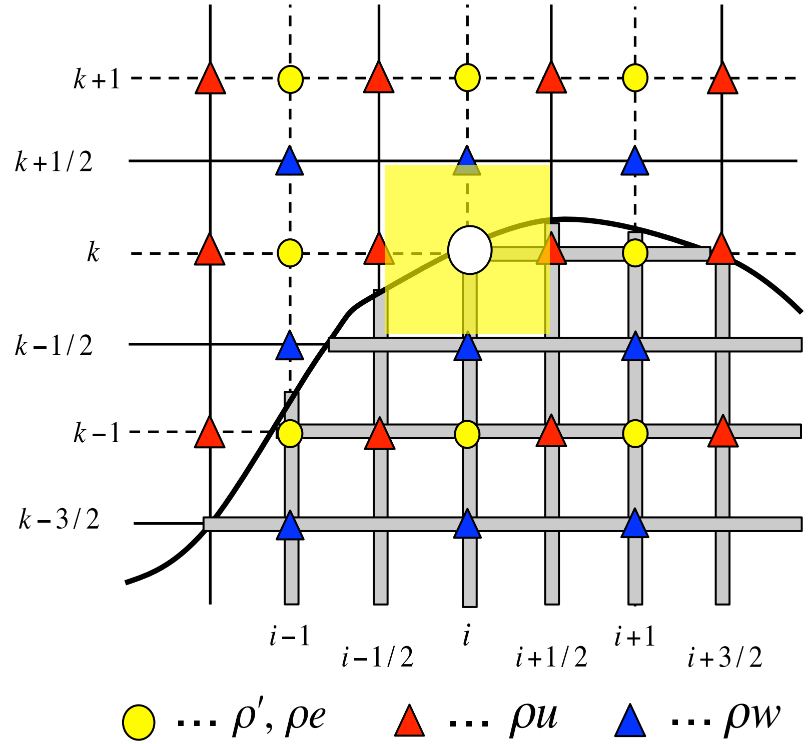


$$u_{i-1/2,k} = 0 \quad if \quad \left(F_z\right)_{i-1/2,k} = \frac{\left(dz'\right)_{i-1/2,k}}{dz} = 0$$

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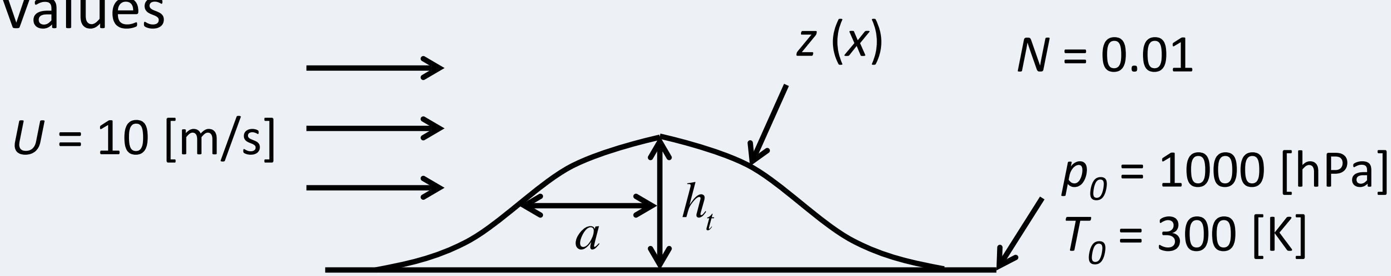
The velocity is set to zero if the side face of a cell defined scalar quantity is consists of thin wall only.

# 4. Mountain wave experiments

## Setup

| | |
|---|---|
| Domain | 4000 [km] × 20 [km] |
| Resolution | 2 [km] × 100 [m] |
| Time step | Δt = 0.25 [s] |
| Boundary condition | Free slip / Periodic |

Initial values




$$z(x) = \frac{h_t a}{x^2 + a^2}$$

$a$ : Half width of the mountain (= 10 [km])

$h_t$ : Height of the mountain

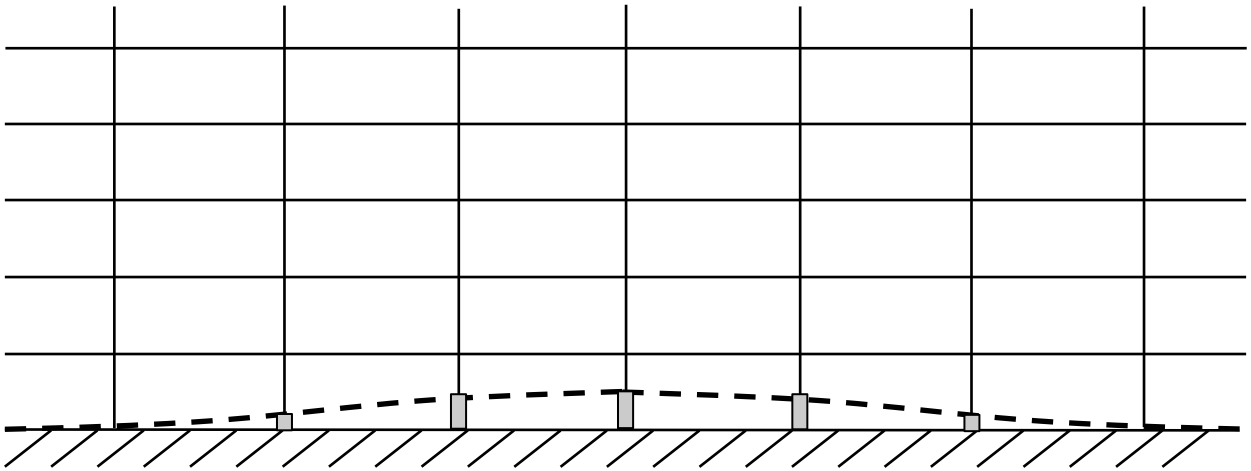
- Sponge layer above 15 [km] height
- 4th order numerical diffusion

# The mountain image created from a thin-wall approximation

$\frac{Nh_t}{U}$ : Scorer number ( The index of nonlinearity of the mountain waves )

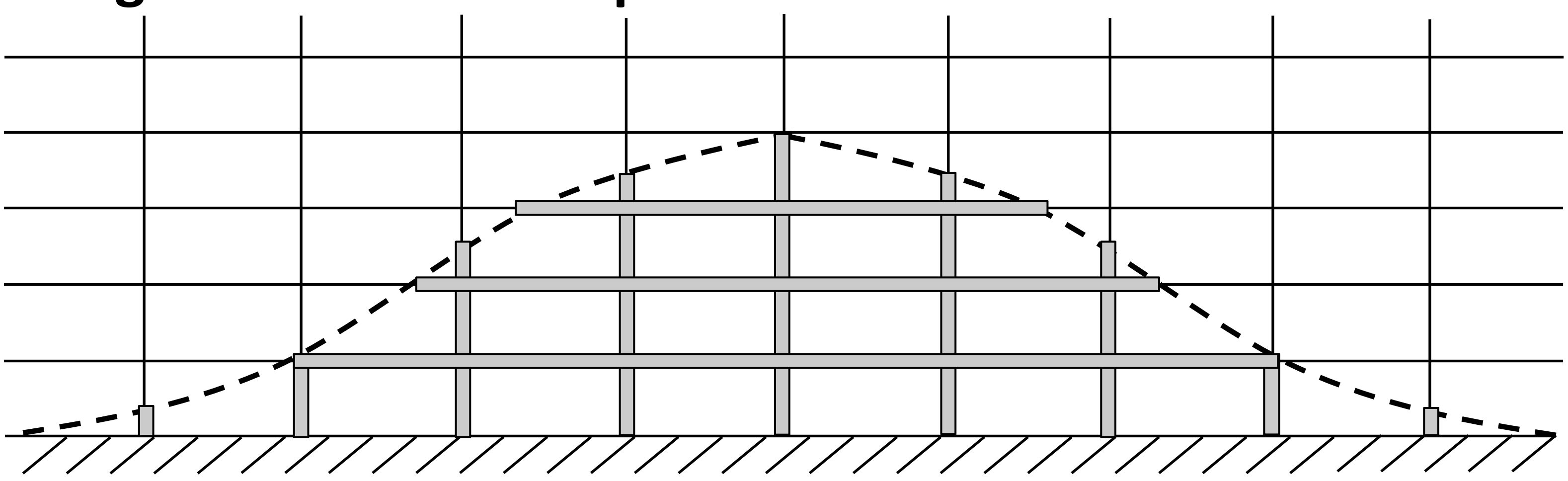
**case A: The height of mountain top is 10m**

$$\frac{Nh_t}{U} = 0.01$$



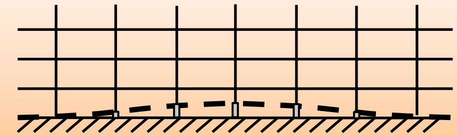
**case B: the height of mountain top is 400m**

$$\frac{Nh_t}{U} = 0.4$$



# A: 10m mountain

$$\frac{Nh_t}{U} = 0.01$$

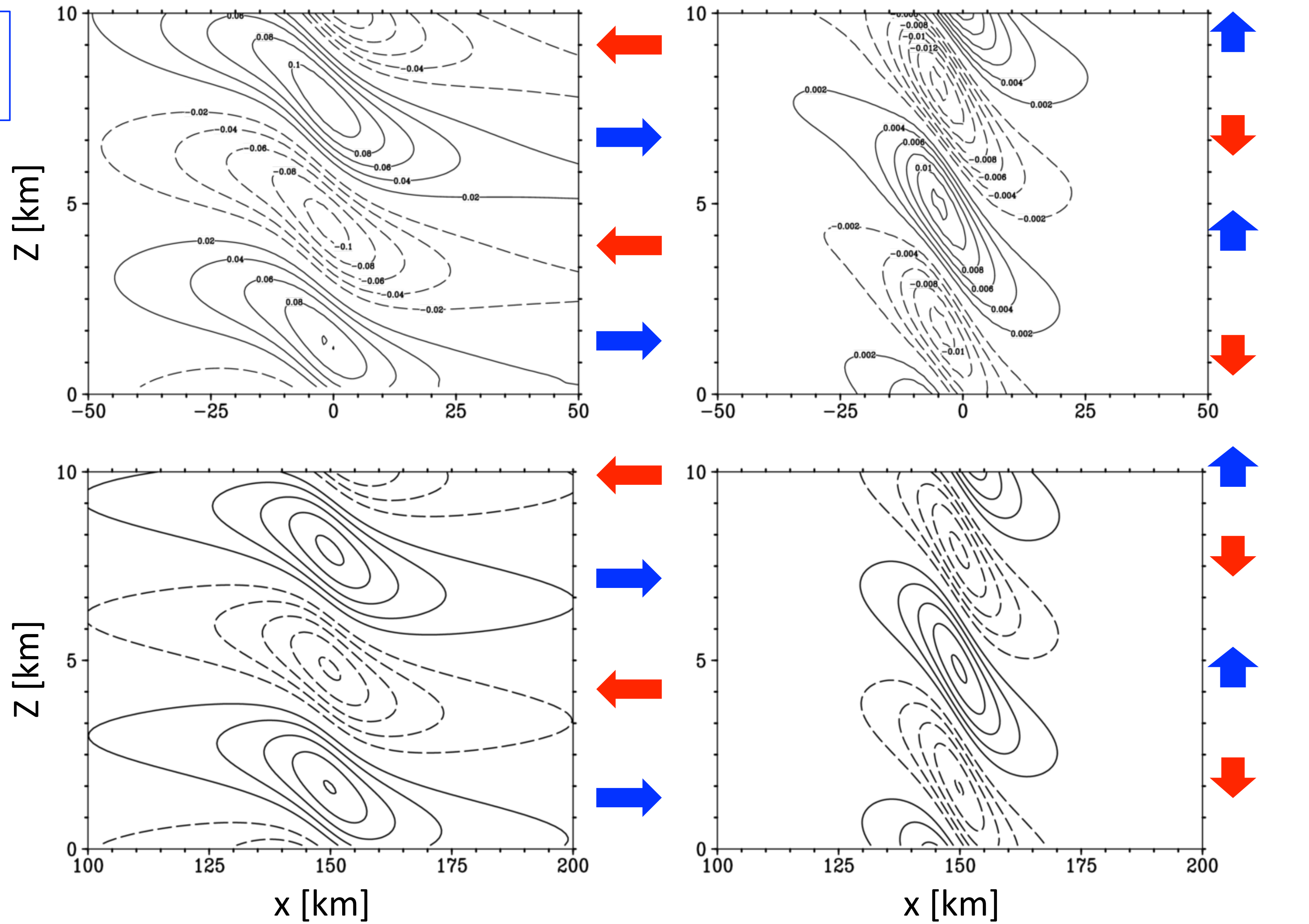


u' (intervals: 0.02 [m s$^{-1}$] )

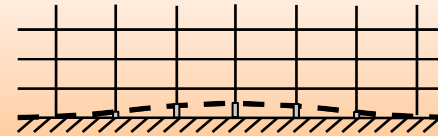
w (intervals: 0.002[m s$^{-1}$] )

**Our model**
**After 12 h**

**Linear theory**



# A: 10m mountain



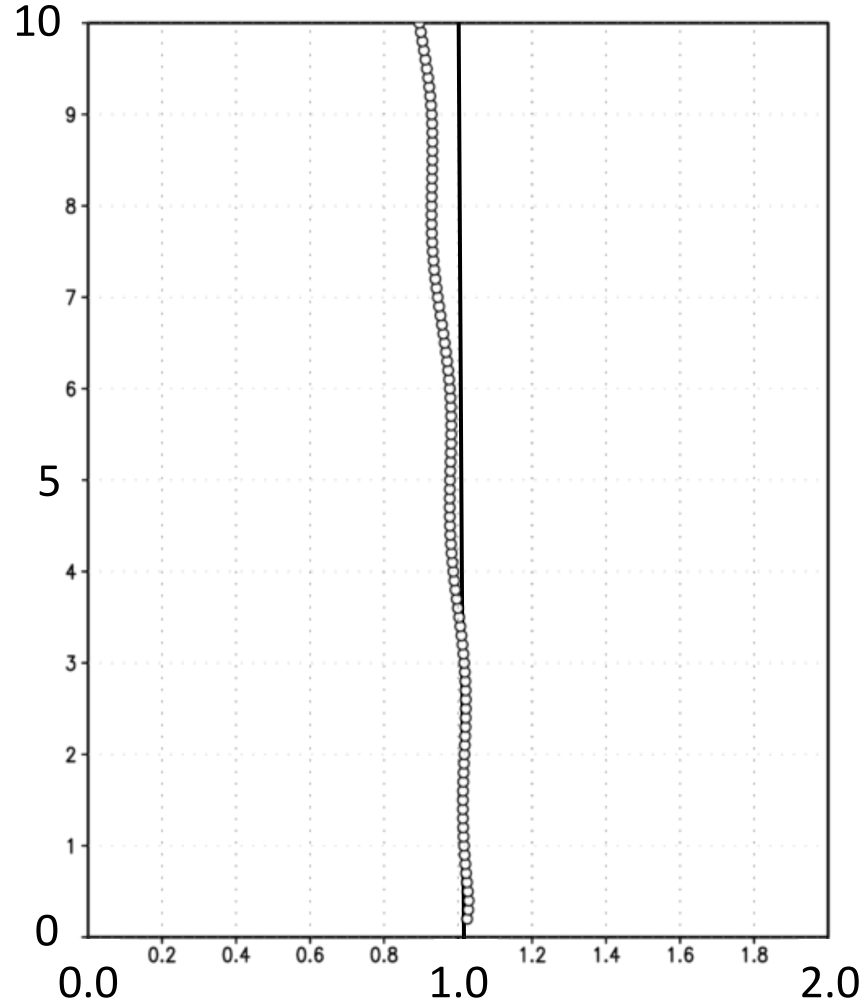
## Vertical transport of horizontal momentum flux

$$MODEL:\ \rho(z)\sum u'w'\ \Delta x$$

$$Linear\ \ theory:\ \frac{\pi}{4}\rho_0 NUh$$

- ✓ The momentum flux in our model is nearly unity.
- ✓ It agrees well with that of the linear theoretical value

After 12 hours



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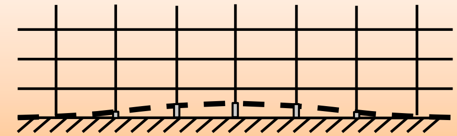


## Horizontal momentum flux budget analysis

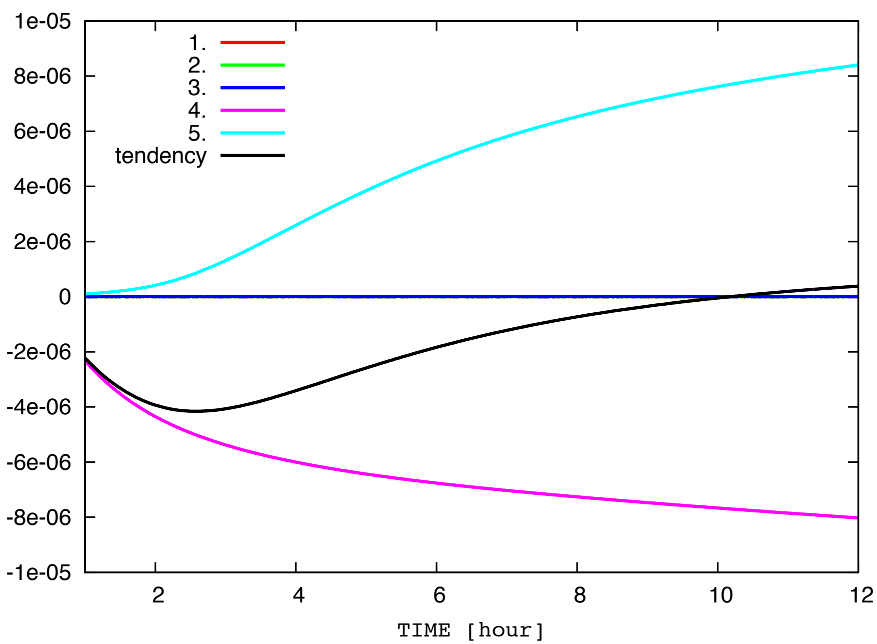
$$\frac{\partial \rho u}{\partial t} = -\frac{\partial F \rho uu}{\partial x} - \frac{\partial F \rho wu}{\partial z} - \frac{\partial p'}{\partial x} + Diff. + Spong.$$

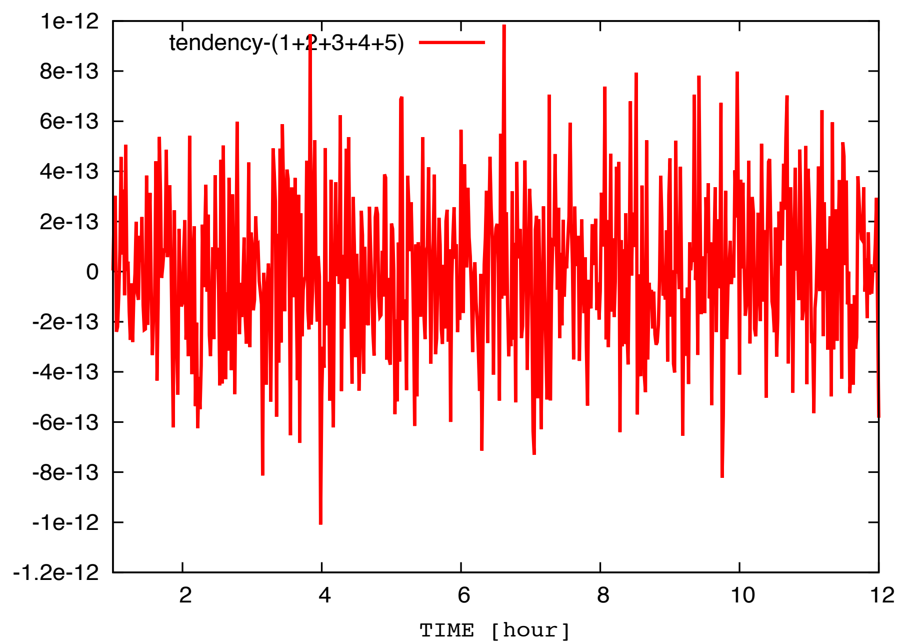
$\boxed{\text{tendency}}$ $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$ $\boxed{5}$

$\boxed{\text{tendency}}$

$- \left( \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4} + \boxed{5} \right)$

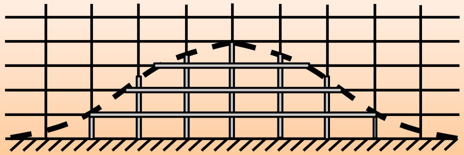
$\sim 10^{-13}$

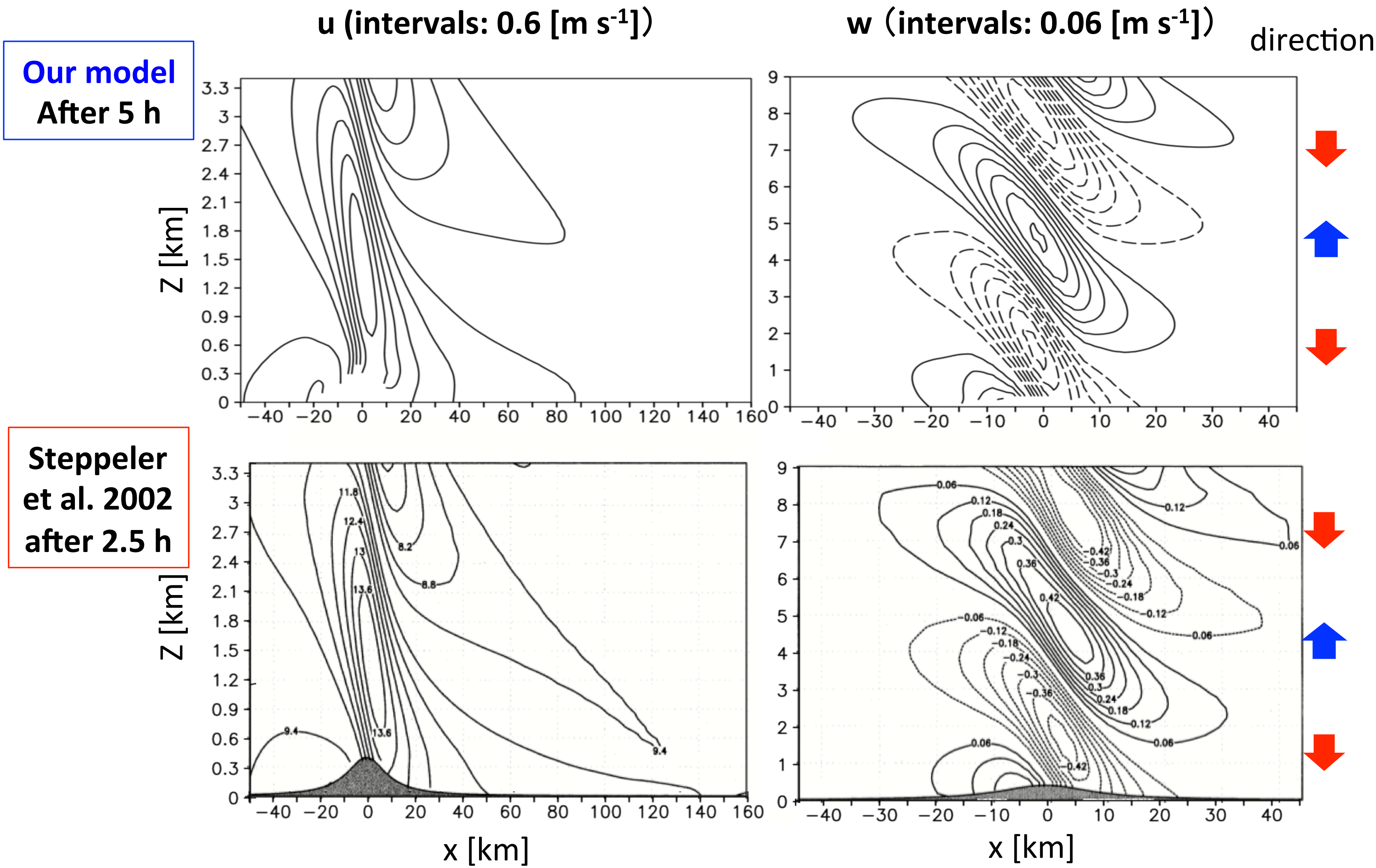


# B: 400m mountain

$$\frac{Nh_t}{U} = 0.4$$





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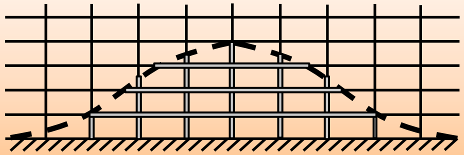
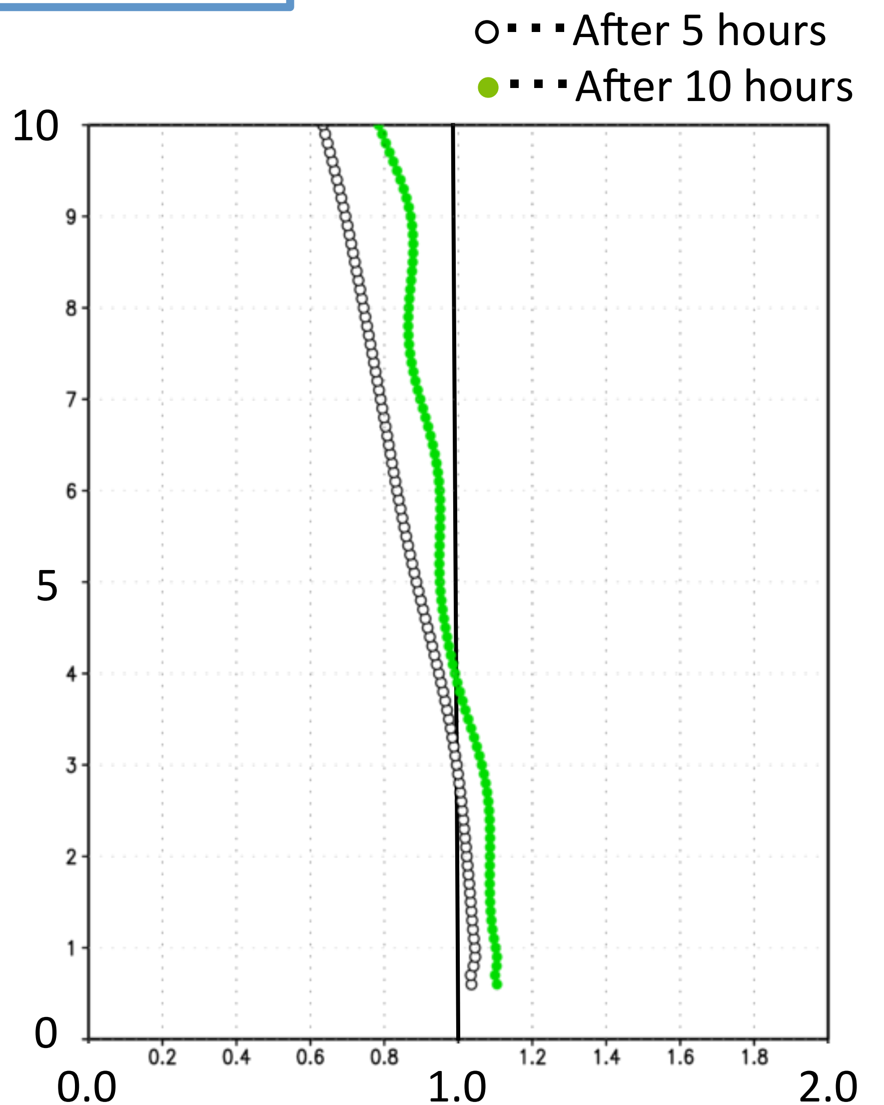


## Vertical transport of horizontal momentum flux

$$MODEL: \rho(z)\sum u'w'\ \Delta x$$

$$Linear\ \ theory: \frac{\pi}{4}\rho_0 NUh$$

- ✓ Nonlinear behavior appears for the result after 10 hours.
- ✓ The momentum fluxes are transported toward the upper layer.



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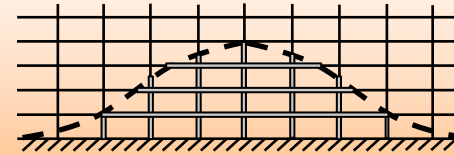


## Horizontal momentum flux budget analysis

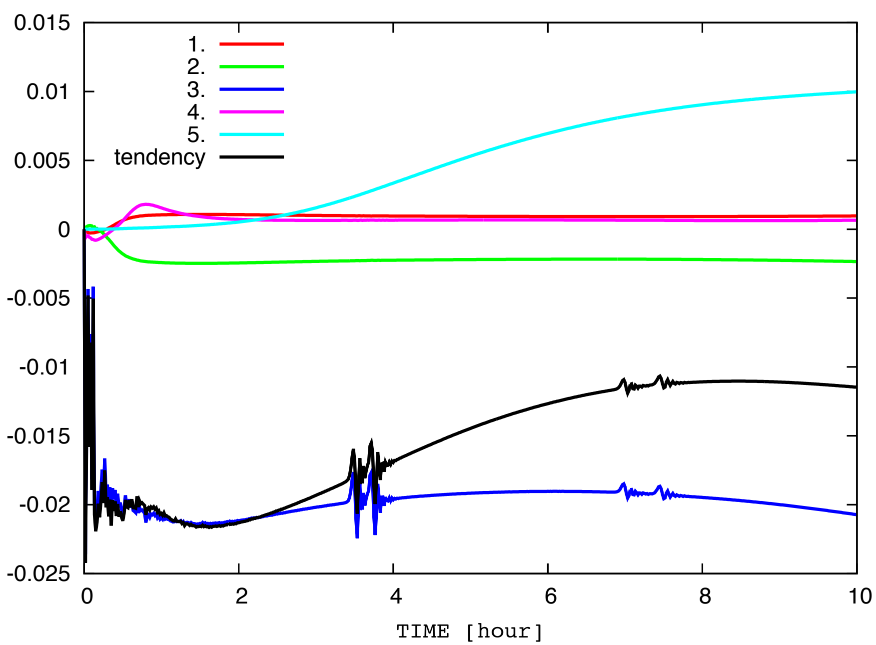
$$\frac{\partial \rho u}{\partial t} = -\frac{\partial F \rho uu}{\partial x} - \frac{\partial F \rho wu}{\partial z} - \frac{\partial p'}{\partial x} + Diff. + Spong.$$

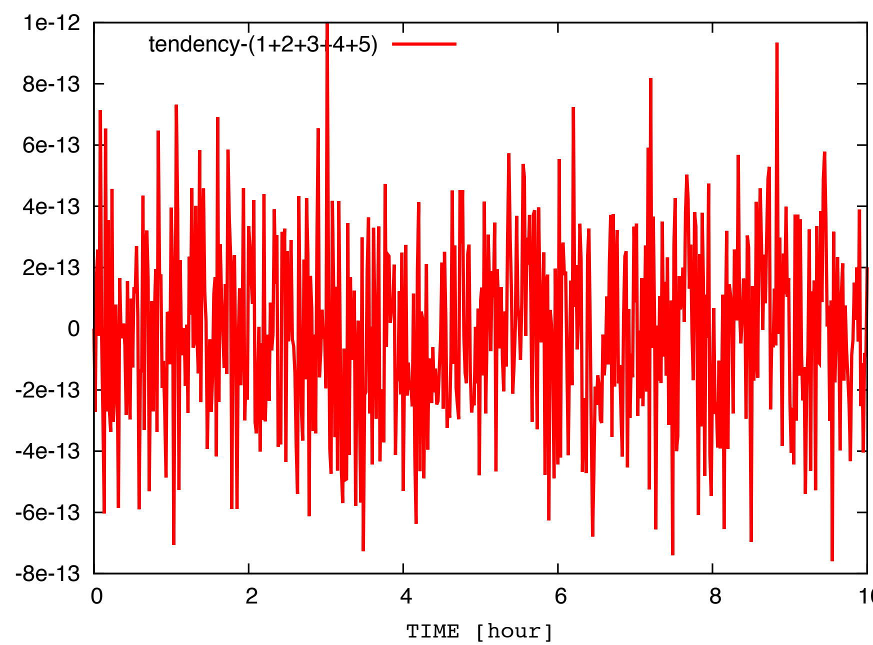
tendency 1 2 3 4 5

tendency

- ( 1 + 2 + 3 + 4 + 5 )

∼$10^{-13}$



# 5. Summary

- ✓ The representation of topography by a thin-wall approximation is implemented to fully compressible flux-form equations.
- ✓ This method addresses the effect of the mountain which height is less than a vertical grid interval.
- ✓ The momentum fluxes are transported toward the upper.
- ✓ The conservation of horizontal momentum flux is confirmed.
- ✓ The future work  is to implement a turbulence scheme to this model.