A Moist Pseudo-Incompressible Model

Warren O'Neill
Professor Rupert Klein

Institut für Mathematik,
Freie Universität,
Berlin

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Outline

Goal

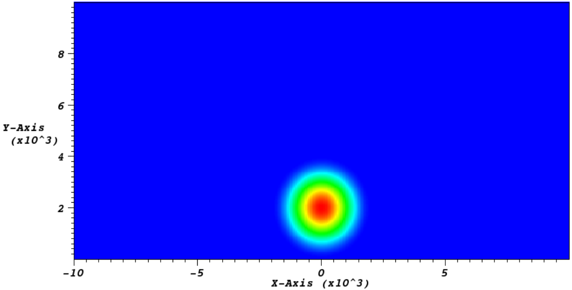


Figure: θ_e inital conditions for the Benchmark test.

Successfully model a hot rising bubble in a moist atmosphere with

- phase changes
- latent heat
- sound waves removed(using P-I approx)

Why pseudo-incompressible?

- filters sound waves allowing longer time steps
- advantages over the anaelastic approximation:
  - allows higher variation in ρ and θ
  - more accurate for small scales e.g. combustion
  - more easily extended to compressible equations
- P-I code currently being developed

The Pseudo-Incompressible approximation

The Compressible Euler Equations

$$
\begin{aligned}
&\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \circ \mathbf{u}) + \nabla p = -g\mathbf{k} \\
&P_t + \nabla \cdot (P \mathbf{u}) = 0
\end{aligned}
$$

where P is an "energy" variable and

$$
P = P(p). \tag{1}
$$

The Pseudo-Incompressible approximation

Assume

$$p = p_0(z) + p'(\mathbf{x}, t)$$
$$\rho = \rho_0(z) + \rho'(\mathbf{x}, t)$$

where $\frac{\partial p_0}{\partial z} = -\rho_0 g$ and $p'/p_0 << 1$.

Now set $p = p_0$ in (1)

$$P = P(p_0) = P_0(z).$$

The Pseudo-Incompressible approximation

The “energy“ equation becomes a divergence constraint

$$(P_0)_t + \nabla \cdot (P_0 \mathbf{u}) = \nabla \cdot (P_0 \mathbf{u}) = 0$$

and the governing equations become

$$\begin{aligned}
&\rho^*_t + \nabla \cdot (\rho^* \mathbf{u}) = 0 \\
&(\rho^* \mathbf{u})_t + \nabla \cdot (\rho^* \mathbf{u} \circ \mathbf{u}) + \nabla p = -\rho^* g \mathbf{k} \\
&\nabla \cdot (P_0 \mathbf{u}) = 0
\end{aligned}$$

where ρ^* is the “pseudo-density”.

Adding moisture

Assumptions

- each state has the same temperature and velocity field
- ignoring: precipitation, ice-phase microphysics, Coriolis force, subgrid-scale turbulence

Adding moisture

Moist compressible equations with bulk thermodynamics

$$
\begin{aligned}
&\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \circ \mathbf{u}) + \nabla p = -g\mathbf{k} \\
&P_t + \nabla \cdot (P\mathbf{u}) = S \\
&\frac{Dq_v}{Dt} = \dot{q}_{cond}, \quad \frac{Dq_c}{Dt} = -\dot{q}_{cond}
\end{aligned}
$$

where $\rho = (\rho_a + \rho_v + \rho_c)$, $q_v = \rho_v/\rho_a$, $q_c = \rho_c/\rho_a$ and S is a latent heat source term.

Equation of state

$$P = P(p, \gamma).$$

New Divergence Constraint

If we use time varying background pressure $p_0(z, t)$ and set $(\gamma = \overline{\gamma})$[1] then P becomes

$$P = P(p_0(z, t), \overline{\gamma}) = P_0(z, t)$$

and we get the divergence constraint

$$\nabla \cdot (P_0 \mathbf{u}) = S - (P_0)_t.$$

The problem now is how do we calculate $(P_0)_t$?

[1] Almgren et al. (2006)

Calculating the time varying background state

- let $w_0(z,t)$ be the vertical velocity field that adjusts the base state and let $\tilde{\mathbf{u}}$ govern the remaining local dynamics

$$\mathbf{u} = w_0\mathbf{k} + \tilde{\mathbf{u}}$$

where $\int_{x_{min}}^{x_{max}} \tilde{w}\,\mathrm{d}x = 0.$

- assume weight of each column remains unchanged in the background state

$$\frac{Dp_0}{Dt} = \frac{\partial p_0}{\partial t} + w_0\frac{\partial p_0}{\partial z} = \frac{\partial p_0^{top}}{\partial t}.$$

Calculating the time varying background state

Making these assumptions we can derive governing equations for the background state

- $\frac{\partial P_0}{\partial t} + \frac{\partial P_0 w_0}{\partial z} = \overline{S}$
- $w_0(z,t) = \int\limits_{z_{min}}^{z} (\overline{S}/P_0 - 1/\overline{\gamma} p_0 \frac{\partial p_0^{top}}{\partial t})\, \mathrm{d}z'$
- $\frac{\partial p_0^{top}}{\partial t} = \frac{\int\limits_{z_{min}}^{z_{max}} \left(\overline{S}/P_0\right) dz}{\int\limits_{z_{min}}^{z_{max}} 1/\overline{\gamma} p_0 \, \mathrm{d}z}$

where $\bar{\cdot} = \frac{1}{L} \int\limits_{x_{min}}^{x_{max}} \cdot \, \mathrm{d}x$.

New governing equations

Mass $\rho^*_t + \nabla \cdot (\rho^* \mathbf{u}) = 0$

Momentum $(\rho^* \mathbf{u})_t + \nabla \cdot (\rho^* \mathbf{u} \circ \mathbf{u})) + \nabla p = -\rho^* g \mathbf{k}$

Microphysics $\frac{Dq_v}{Dt} = \dot{q}_{cond}, \quad \frac{Dq_c}{Dt} = -\dot{q}_{cond}$

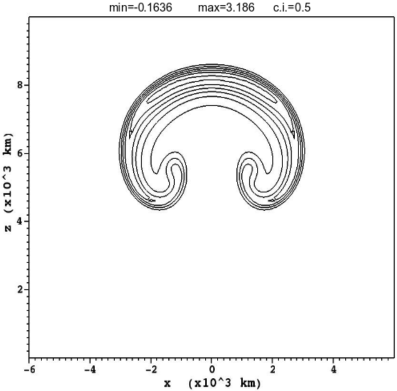
Base-state updates
$$\begin{cases} \frac{\partial P_0}{\partial t} + \frac{\partial P_0 w_0}{\partial z} = \overline{S} \\ w_0(z,t) = \int_{z_{min}}^{z} (\overline{S}/P_0 - 1/\overline{\gamma} p_0 \frac{\partial p_0^{top}}{\partial t}) \, \mathrm{d}z' \end{cases}$$

Divergence constraint $\nabla \cdot (P_0 \mathbf{u}) = S - (P_0)_t$

Benchmark Test-Case

Video

Benchmark Test-Case



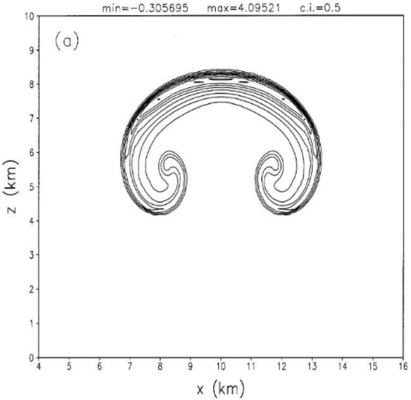


Figure: Contour plots of θ_e.

Benchmark Test-Case

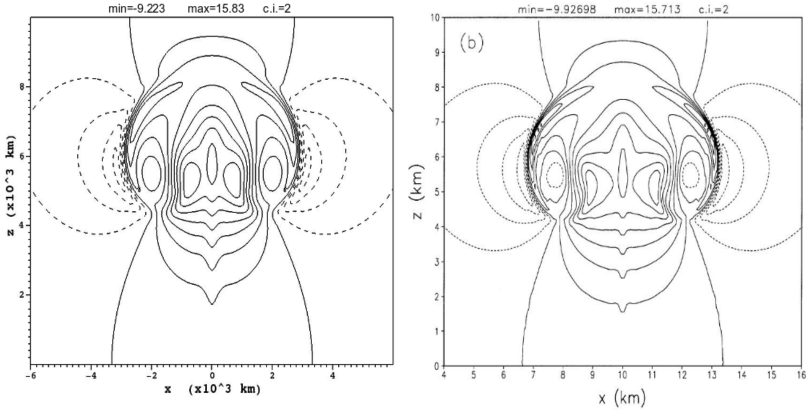


Figure: Contour plots of the vertical velocity.

Conclusion

A working moist pseudo-incompressible model which is

- comparible in accuracy to a compressible model for the benchmark test-case
- valid for a wider range of values than the anelastic model
- “easily” extendable to a compressible model

Future plans

- parallelisation and mesh-refinement (already implemented in dry case)
- add precipitation
- other test cases e.g. the more realistic test-case of Klassen & Clark 1985, squall-lines....

Thank you for your attention