Testing New Approaches for LBC formulation in Spectral LAM

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Outline

1. Introduction
   - Spectral formulations with Time-dependent LBCs
   - Boundary effects
2. Well-posed Transparent LBC
   - Well-posed LBC in 2 TL SISL scheme
   - Well-posed LBC externally treated from the Dynamics
3. New absorbing boundary Layers
   - Perfectly Match Layer technique
   - Encouraging results
4. Prospects and concluding remarks

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Spectral Formulations : Periodicity issue

Pertubation Method : Hoyer (1987)

- Equations of motion are pertubated around the large scale solution,
- The deviation are "relaxed" towards zero at boundary.

Sinusoidal Subtracted Expanded Method : Tatsumi (1986)

- The solution is seeked as the sum of a time-dependent sinusiodal function and sine series with rigid wall-boundary condition.
- The additional time-dependent function are introduced to satisfy time-dependent LBC.

Fourier Extention Method : Haugen and Machenhauer (1992)

- The fields are extended into an artificial zone (so-called E-zone) by means of Spline functions in order to fullfill periodicity and continuity of the first order derivatives.

Spectral Formulations : Coupling issue

Davies Flow relaxation scheme

- All the pronostic variables are relaxed towards the large scale solution into a additional zone adjacent to the boundary as :

$$\phi_i = (1 - \alpha_i)\,\phi_i^g + \alpha_i\,\phi_i^h$$

- Philosophy : Over-specifiy and Damp the resulting noises.

Weak points :

(i) Over-specification : it may introduce unnecessary errors ,

(ii) It may destroy mass conservation and geostrophic balance,

(iii) It is not fully transparent (and/or absorbing) layer

(iv) It has not been built in a rigourous mathematical grounds

Boundary effects

Does well-posed LBC is a suitable strategy in Fourier SLAM framework ?

Consider the linearized one dimensional advection equation :

$$\frac{\partial \psi}{\partial t} + U \frac{\partial \psi}{\partial x} = 0$$

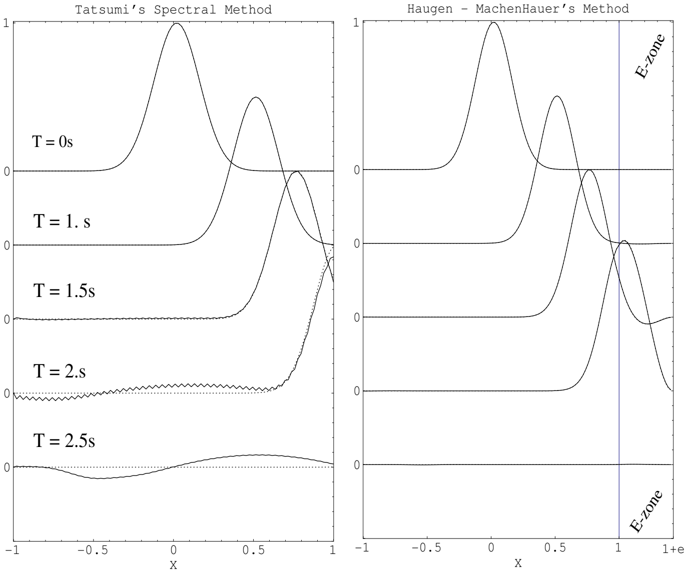
, Assume that $U > 0$, ψ is to be determine throughtout the limited area $-1 \leq x \leq 1$. At any time the solution is dicted its upwind history, so that :

$$\psi(x, t) = \psi(x - Ut, t).$$

To yield a well-posed boundary condition :

- $\psi(-1, L)$ must be externally supplied.
- $\psi(1, L)$ must be determined by the gorverning equations

Numerical Test : Exiting Bell-shape

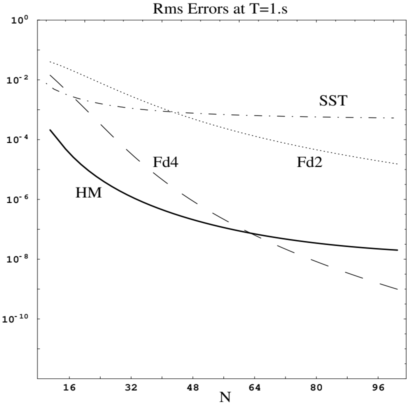


Boundary effects

- In Tatsumi Method, Gibbs phenomenon causes considerable distorsion of the solution. It makes Well posed LBC inappropriated for this method.

Fourier Extension (HM)

- No adverse effects related to Gibbs phenomenon
- Slow convergence rate, but acceptable for high resolution modelling



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Model equations

One dimensional flow

Let us consider the linearized 1D shallow water equations :

$$\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x} + \bar{\phi}\,\frac{\partial \Phi}{\partial x} - \bar{f}\,v = 0 \quad (1)$$

$$\frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial x} + \bar{f}\,u = 0 \quad (2)$$

$$\frac{\partial \Phi}{\partial t} + \bar{u}\frac{\partial \Phi}{\partial x} + \frac{\partial u}{\partial x} = 0 \quad (3)$$

We assume that $\bar{u} > 0$ and $\sqrt{p\bar{h}i} = 300ms^{-1}$. These equations are to be solved over the limited area $0 \leq x \leq L$.

Well-posed Characteristics LBC

Using the transformation $p = u + \Phi\sqrt{\bar{\phi}}$ and $q = u - \Phi\sqrt{\bar{\phi}}$, the equations $(1) - (3)$ becomes :

$$\frac{\partial p}{\partial t} + (\bar{u} + \sqrt{\bar{\phi}})\frac{\partial p}{\partial x} = \bar{f}\, v \tag{4}$$

$$\frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial x} = -\frac{\bar{f}}{2}\,(p+q) \tag{5}$$

$$\frac{\partial q}{\partial t} + (\bar{u} - \sqrt{\bar{\phi}})\frac{\partial q}{\partial x} = \bar{f}\, v \tag{6}$$

- If $\bar{u} > \sqrt{\bar{\phi}}$, then we must prescribe p, v, and q at $x = 0$, no fields are to be supplied at $x = L$ boundary.
- If $\bar{u} < \sqrt{\bar{\phi}}$, then we must prescribe only p and v at $x = 0$, q has to be prescribed at $x = L$.

Well-posed LBC for Semi-Lagrangian schemes

Let us perform a two time-level SLSI discretization :

$$\left(I - \frac{\Delta t}{2}\bar{\mathcal{L}}\right)\mathbf{\Psi}_A^+ \;=\; \left(I + \frac{\Delta t}{2}\bar{\mathcal{L}}\right)\mathbf{\Psi}_O^0$$

with,

$$\mathcal{L} \;=\; \begin{pmatrix} 0 & \bar{f} & -\bar{\phi}\,\frac{\partial}{\partial x} \\ -\bar{f} & 0 & 0 \\ -\frac{\partial}{\partial x} & 0 & 0 \end{pmatrix}.$$

A denotes arrival point and O corresponds to the departure point. This SI scheme leads to the resolution of the so-called Helmholtz equation.

How does the well-posed LBC has to be incorporated to yield a stable and accurate solution ?

Well-posed LBC for SISL schemes

McDonald's Idea for Finite-difference discretization

- LBCs are to be inserted into the implicit part of the SI correction
- Useful external points are extrapolated from the interior

Spectral discretization constraints

- The implicit operator must have horizontally homogeneous coefficients
- The explicit RHS (Right and Side) terms have to be periodic.

- LBCs are to be inserted at the beginning of the time step in a explicit way
- the resulting RHS explicit terms have to be periodically extended at each time step.

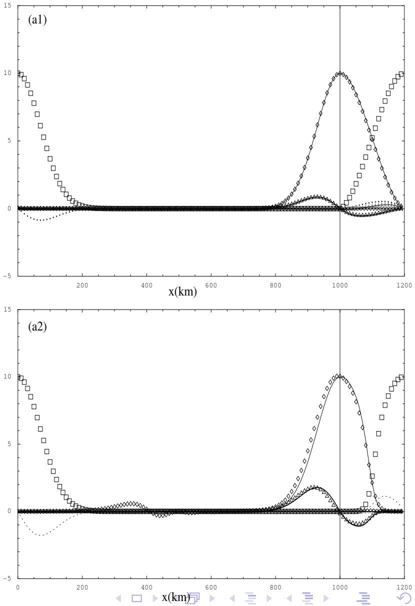
Explicit Well-posed LBC

- Explicit LBC treatment causes severe restriction on the time step so that :

$$C_{\bar{\phi}} = \frac{\Delta t\sqrt{\bar{\phi}}}{\Delta x} < 2.$$

a1 $\Delta t = 50s$, $\bar{u} = 100ms^{-1}$,$\Delta x = 10km$, $C_{\bar{u}} < 1$ solution is stable and accurate

a2 $\bar{u} = 250ms^{-1}$, $C_{\bar{u}} = 1.5$ the solution is inaccurate.



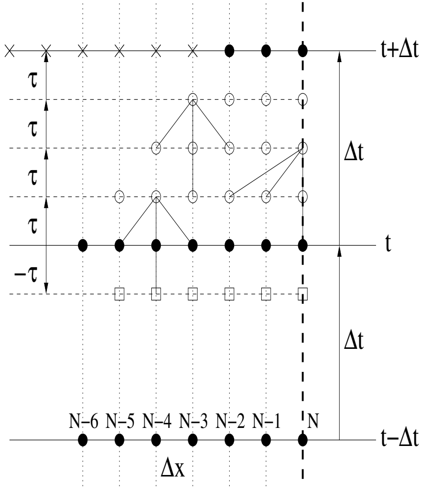
Well-posed LBC externally treated

Sub-stepping idea

- fractionned the time step Δt in small τ with $n_\tau = \Delta/\tau$.
- Computation a guess $\tilde{\mathbf{\Psi}}_{LBC}$ with a explicit scheme over

$$N_{sub} = n_\tau + N_{buf}$$

- Built a FD implicit operator $(I - \delta t L)\tilde{\mathbf{\Psi}}_{LBC}$ thanks to the remaining N_{buf} grid point.

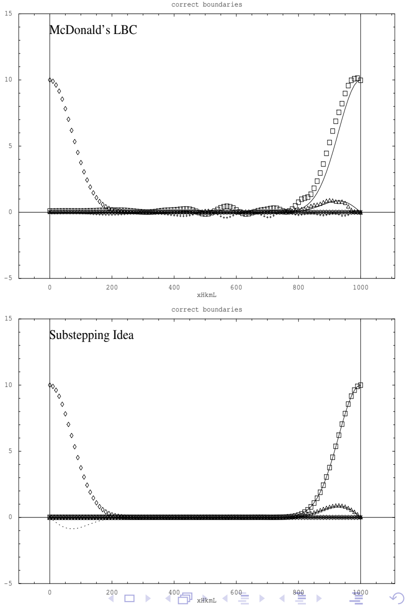


$$\tilde{\mathbf{\Psi}}^{+}_{LBC} = \mathcal{M}^{(n_\tau)}_{\tau} \circ \mathcal{M}^{(n_\tau - 1)}_{\tau} \circ \cdots \circ \mathcal{M}^{(2)}_{\tau} \circ \mathcal{M}^{(1)}_{\tau} \left(\mathbf{\Psi}^{0}_{g}, \mathbf{\Psi}^{+}_{h}\right)$$

Well-posed LBC externally treated

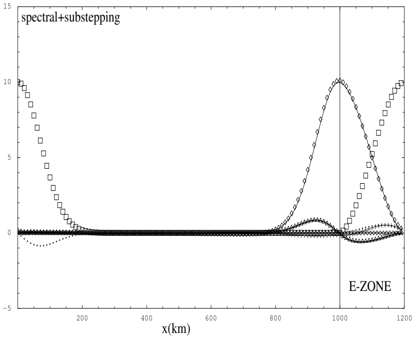
Does it work for FD discretization ?

- $\Delta t = 400$, $\bar{u} = 100 ms^{-1}$, $\delta x = 10km$
- For larger advectif CLF, McDonald LBC treatment is inaccurate
- $\tau = 19,7s, Nsub = 2n_\tau = 32$
- Substepping provides stable and accurate solution
- It's very expensive



Spectral+LBC sub-stepping idea

- $\Delta t = 100s$, $\bar{u} = 150ms^{-1}$, $\sqrt{\bar{\phi}} = 300ms^{-1}$, $\tau = 20s$, $N_{sub} = 8$. so, $C_{\bar{\phi}} = 3$ and $C_{\bar{u}} = 1.5$.
- we also use a predictor/corrector SL explicit scheme



Accuracy issue

- for larger time step, we got stability, but the accuracy of the solution is threathened. One suspect some lack of compatibility between spectral and finite difference implicit operator.

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Some historical issues

- spiltted PML has been invented by Bérenger (1994) for electromagnetic issue
- Navon et *al.* (2004), was first to introduce this technique for meteorological context. But their PML scheme is untractable for both spectral and SI scheme.
- Hu(2006) has recently developped a new unsplitted well-posed PML for Non-linear Euler-Equations.

Does Hu's PML formulation can be Matched to the NWP framework (Spectral SI) ?

Hu's formulation for 1D shallow water

$$\begin{aligned}
\frac{du}{dt} + \bar{\phi}\frac{\partial \Phi}{\partial x} - \bar{f}\, v &= -\sigma(x)\left[(1+\bar{u}\mu)\tilde{u} + \left(\bar{\phi}(\mu\tilde{\Phi} + a_{\Phi}) - \bar{f}\, a_v\right)\right] & (7)\\
\frac{dv}{dt} + \bar{f}\, u &= -\sigma(x)\left[(1+\bar{u}\mu)\tilde{v} + \bar{f}\, a_u\right] & (8)\\
\frac{d\Phi}{dt} + \frac{\partial u}{\partial x} &= -\sigma(x)\left[(1+\bar{u}\mu)\tilde{\Phi} + (\mu\tilde{u} + a_u)\right] & (9)
\end{aligned}$$

Auxiliary variables $\mathbf{a} = (a_{\Phi}, a_u, a_v)$ are necessary inside the PML zone to acheive stability :

$$\frac{\partial \mathbf{a}}{\partial t} = \tilde{\mathbf{\Psi}}$$

and *mu* is a stability parameter :$\mu = \bar{u}\,(\bar{\phi} - \bar{u}^2)^{-1}$

PML SI scheme

Let consider the 3TL SI Eulerian scheme :

$$\left(I - \Delta t\, \bar{\mathcal{L}}\right)\ \mathbf{\Psi}^{+} \quad = \quad \mathbf{\Psi}_{exp} + \Delta t\, \bar{\mathcal{L}}\ \left(\mathbf{\Psi}^{-} - 2\mathbf{\Psi}^{0}\right)$$

To insert the PML absorbing terms, we follow the Radnoti's relaxation scheme :

$$\left(I - \Delta t\, \bar{\mathcal{L}}\right)\ \mathbf{\Psi}_i^{+} \quad = \quad (1-\gamma_i)\mathcal{R}_i^{g} + \gamma_i \left(I - \Delta t\, \bar{\mathcal{L}}\right)\ \mathbf{\Psi}_i^{h,+} + \gamma_i \mathcal{R}_i^{pml} \quad (10)$$

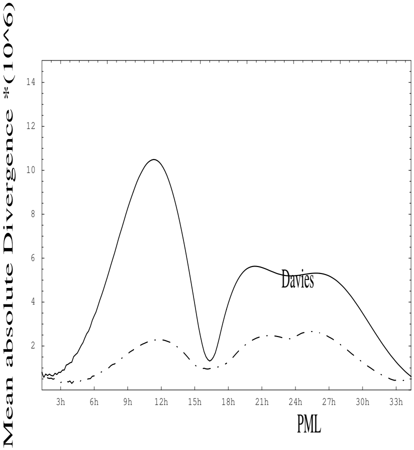
where

$$\gamma_i = \frac{2\Delta t \sigma_i}{1 + 2\Delta t \sigma_i}$$

PML numerical test

Bell-shape exitng the area

- $\Delta t = 200s$, $\bar{u} = 12.5ms^{-1}$, $\Delta x = 10km$
- Stable and accurate solution
- Less gravity waves surplus in the PML experiment
- $Rms(\Phi)_{dav} = 0.08m^2s^{-2}$, $Rms(\Phi)_{PML} = 0.02m^2s^{-2}$.



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Prospects and concluding remarks

- Design a Well-posed LBC in Spectral LAM implies that we seek coupling process as an external trend, treated independently from the Dynamics.
- If well-posed LBC reveals inappropriate for SLAM, it's not a reason for depressing. Some new way of constructing abrbing boundary layer in a more mathematical grounds than Davies' Newtonian relaxation, has been succefully tested with very encouraging results.